Explorations in Recursion with John Pell and the Pell Sequence

Recurrence Relations and their Explicit Formulas

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Abstract

The overall objective of my Math 501 research project is to create explicit closed-form formulas from homogeneous and non-homogeneous linear recurrence relations using four techniques: (i) guess and check with induction, (ii) the characteristic polynomial, (iii) generating functions, and (iv) linear algebra. Recurrence relations are a central mathematical topic, frequently taught in courses such as Discrete or Finite Math, Combinatorics, Number Theory, and Computer Science. After presenting the four techniques for solving recurrence relations, I include some background information on each topic as well as a few examples for each.

Another objective of the project is to explore the Pell Sequence, also known as the Pell Numbers. Like other famous sequences, such as the Fibonacci or the Lucas Sequence, the Pell Sequence has some interesting properties of its own (Bicknell, 1975). Within the study of the Pell Sequence other topics that will be explored include: (i) the "obscure" mathematician John Pell (Webster, 2006); (ii) the history, properties, and identities of the Pell Sequence; (iii) the solution of an explicit formula from the recurrence relation of the Pell Sequence with the four techniques; and (iv) some proofs of an alternate closed-form version of the Pell Sequence.

Finally, I will give an explanation and summary of the curriculum that may be used in a high school or college level classroom. In this portion of the project, there are materials for an instructor and students to use, including: lessons for students,

lessons for instructors with solutions, lesson plans for instructors, reflections and summaries of the students' lessons.

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Chapter 1

An Introduction to Recurrence Relations

Recursion is a process in which each step relies on a previous step or set of steps. Sequences of numbers can be defined recursively by means of such a relationship, often given by an equation called a *recurrence relation*. Recall that a *sequence* is a function whose domain is some set of integers (usually the natural numbers N) and whose range is a set of real numbers (Goodaire & Parmenter, 2006, p.160). The numbers in the list (the range of the function) are called the *terms* of the sequence (Goodaire & Parmenter, 2006, p.160). For example, the sequence of numbers

$$0,1,2,5,12,29,70,169,408,\ldots$$

Can be defined with the recurrence relation

$$p_n = 2p_{n-1} + p_{n-2}$$

Where $p_0 = 0$ and $p_1 = 1$ for all $n \geq 2$ (Goodaire & Parmenter, 2006). The requirements that $p_0 = 0, p_1 = 1$ are known as the *initial conditions* of the recurrence relation. The rest of the terms in the sequence can be calculated successively using the rule above.

Sometimes, recurrence relations may be expressed with what is known as a *closed-form* formula (also an *explicit* formula, or a "solution") (Goodaire & Parmenter, 2006). For example, a solution to the recurrence relation above is

$$p_n = \left(\frac{\sqrt{2}}{4}\right)\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right]$$

For all $n \geq 0$ (Gullberg, 1997, p. 288). Variety abounds in this subject -- multiple recurrence relations and multiple explicit formulas may be given to describe any specific sequence. In this project, we explore the mathematics of finding such expressions for recursively-defined sequences.

Two common types of number sequences are *arithmetic* and *geometric* sequences. A term is created in an arithmetic sequence by adding the same fixed number (known as the *common difference*) to the previous term (Goodaire & Parmenter, 2006). For example,

$$-4, -2, 0, 2, 4, \ldots..$$

Is an arithmetic sequence with a common difference of 2. An arithmetic sequence with the first term a and the common difference d can be defined recursively by $a_1 = a$ and for $n \geq 1$,

$$a_{n+1} = a_n + d$$

(Goodaire & Parmenter, 2006, p.162).

The solution to an arithmetic sequence is

$$a_n = a + (n-1)d$$

Where $n \geq 1$ (Goodaire & Parmenter, 2006, p.162).

A geometric sequence, on the other hand, occurs when each term is created by multiplying the previous term with the same number (known as the *common ratio*) (Goodaire & Parmenter, 2006). For example,

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$$

Is a geometric sequence with a common ratio of $\frac{1}{3}$. A geometric sequence with the first term a and the common ratio r can be defined recursively by $a_1 = a$ and for $n \geq 1$,

$$a_{n+1} = ra_n$$

(Goodaire & Parmenter, 2006, p.163).

The solution to such a geometric sequence is

$$a_n = ar^{n-1}$$

Where $n \geq 1$ (Goodaire & Parmenter, 2006, p.163).

Some authors use the terms *difference equation* and *recurrence relation* interchangeably (Balakrishnan, 1991, p.95). While there are no general methods for solving all recurrence relations (Balakrishnan, 1991, p.95), in the next chapter, I will demonstrate four different techniques for solving certain basic recurrence relations.

Chapter 2

Four Techniques for an Explicit Formula

2.1: Guess and Check with the Principle of Mathematical Induction

In this section I will use the Principle of Mathematical Induction with recurrence relations to prove that a conjectured solution, or explicit formula, for the recurrence relation is indeed correct. Mathematical Induction is a way to establish the truth of a statement about all the natural numbers or, sometimes, all sufficiently large integers (Goodaire & Parmenter, 2006, p.147). The principle of mathematical induction states:

Given a statement $P(n)$ concerning the integer n, suppose

1. $P(n_0)$ Is true for some particular integer n_0.
2. If $k \geq n_0$ is an integer and $P(k)$ is true, then $P(k+1)$ is true.
   Then, $P(n)$ is true for all integers $n \geq n_0$.

(Goodaire & Parmenter, 2006, p.149).

In Step 2, the assumption that P is true for some particular integer is known as the *induction hypothesis* (Goodaire & Parmenter, 2006, p.149).

Given a recurrence relation, I will show the first few terms of the sequence, guess an explicit formula or a "solution" for the sequence, and show by the Principle of Mathematical Induction that this explicit formula is valid.

Example 1. Consider the sequence defined by $a_1 = 1$ and, for $n \geq 1$, $a_n = 2a_{n-1} + 1$

(Goodaire & Parmenter, 2006, p.167).

The first few terms of this sequence can be computed as follows:

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= 2a_{2-1} + 1 = 2a_1 + 1 = 2(1) + 1 = 3 \\
a_3 &= 2a_{3-1} + 1 = 2a_2 + 1 = 2(3) + 1 = 7 \\
a_4 &= 2a_{4-1} + 1 = 2a_3 + 1 = 2(7) + 1 = 15 \\
a_5 &= 2a_{5-1} + 1 = 2a_4 + 1 = 2(15) + 1 = 31 \\
a_6 &= 2a_{6-1} + 1 = 2a_5 + 1 = 2(31) + 1 = 63
\end{aligned}
$$

From this data, we can notice the following pattern and guess a formula:

$$
\begin{aligned}
a_1 &= 2^1 - 1 = 1 \\
a_2 &= 2^2 - 1 = 3 \\
a_3 &= 2^3 - 1 = 7 \\
a_4 &= 2^4 - 1 = 15 \\
a_5 &= 2^5 - 1 = 31 \\
a_6 &= 2^6 - 1 = 63 \\
\therefore a_n &= 2^n - 1, \forall n \geq 1
\end{aligned}
$$

We now use induction to prove that the conjecture $a_n = 2^n - 1$ holds for all $n \geq 1$.

Proof:

(i) Base case:

For $n = 1 \rightarrow a_n = 2^n - 1 \rightarrow a_1 = 2^1 - 1 = 1$. Check

(ii) Induction step:

Assume $a_n = 2^n - 1$ is true then $a_{n+1} = 2^{n+1} - 1$ is true. Then

$$
\begin{aligned}
&a_{n+1} = 2a_{(n+1)-1} + 1 \rightarrow 2a_n + 1 \rightarrow 2(2^n - 1) + 1 \\
&\rightarrow 2^{n+1} - 2 + 1 \rightarrow 2^{n+1} - 1
\end{aligned}
$$

$\therefore$ By induction $a_n = 2^n - 1$ holds $\forall n \geq 1$.

Example 2. Consider the sequence defined by $a_0 = 2, a_1 = 3$ and $a_n = 3a_{n-1} - 2a_{n-2}$

For $n \geq 2$ (Goodaire & Parmenter, 2006, p.167).

The first few terms of this sequence can be computed as follows:

$$\begin{aligned}
a_0 &= 2 \\
a_1 &= 3 \\
a_2 &= 3a_1 - 2a_0 = 3(3) - 2(2) = 5 \\
a_3 &= 3a_2 - 2a_1 = 3(5) - 2(3) = 9 \\
a_4 &= 3a_3 - 2a_2 = 3(9) - 2(5) = 17 \\
a_5 &= 3a_4 - 2a_3 = 3(17) - 2(9) = 33
\end{aligned}$$

From this data, we can notice the following pattern and guess a formula:

$$\begin{aligned}
a_0 &= 2^0 + 1 = 1 \\
a_1 &= 2^1 + 1 = 3 \\
a_2 &= 2^2 + 1 = 5 \\
a_3 &= 2^3 + 1 = 9 \\
a_4 &= 2^4 + 1 = 17 \\
a_5 &= 2^5 + 1 = 33 \\
\therefore a_n &= 2^n + 1, \forall n \geq 0
\end{aligned}$$

We now use induction to prove that the formula $a_n = 2^n + 1$ holds for all $n \geq 0$.

Proof:

(i) Base cases:

For $n = 0 \rightarrow a_n = 2^n + 1 \rightarrow a_0 = 2^0 + 1 = 2$. Check
For $n = 1 \rightarrow a_n = 2^n + 1 \rightarrow a_1 = 2^1 + 1 = 3$. Check.

(ii) Induction step:

Assume $a_n = 2^n + 1$ is true and $a_{n-1} = 2^{n-1} + 1$ is true, then the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ becomes:

$$a_{n+1} = 3a_{(n+1)-1} - 2a_{(n+1)-2} \rightarrow 3a_n - 2a_{n-1}$$

$$3(2^n + 1) - 2(2^{n-1} + 1) \rightarrow 3(2^n + 1) - 2^n - 2 \rightarrow$$

$$3(2^n + 1) - (2^n + 1) - 1 \rightarrow (2^n + 1)(3 - 1) - 1 \rightarrow n \geq 0.$$

$$(2^n + 1)(2) - 1 \rightarrow 2^{n+1} + 2 - 1 \rightarrow 2^{n+1} + 1$$

$$\therefore a_n = 2^n + 1$$

So the formula holds for $\forall\ n \geq 0$.

2.2: The Characteristic Polynomial

The method of characteristic polynomials is commonly used to solve a recurrence relation when it takes the form

$$a_n = ra_{n-1} + sa_{n-2} + f(n)$$

Where r and s are constants, and $f(n)$ is some function of n (Goodaire & Parmenter, 2006, p.170). If $f(n) = 0$, then this type is called a *second order linear recurrence relation with constant coefficients* and the relation is called *homogeneous* (Goodaire & Parmenter, 2006, p.170). If $f(n) \neq 0$ then the recurrence relation is *non-homogeneous*. Such a recurrence relation is *second order* if it defines a_n as a function of the two terms preceding it (Goodaire & Parmenter, 2006, p.170). It is *linear* because a_{n-1} and a_{n-2} are not multiplied together and they both occur to the first power, and, clearly, they have *constant coefficients* (Goodaire & Parmenter, 2006, p.171).

We can rewrite the homogeneous recurrence relation where $f(n) = 0$, as

$$a_n - ra_{n-1} - sa_{n-2} = 0$$

(Goodaire & Parmenter, 2006, p.170).

We associate this recurrence relation with the quadratic polynomial

$$x^2 - rx - s$$

(Goodaire & Parmenter, 2006, p.171).

This polynomial is the *characteristic polynomial* of the recurrence (Goodaire & Parmenter, 2006, p.171). In order to solve such recurrence relations, suppose that x_1 and x_2 are roots of the polynomial $x^2 - rx - s$, then the solution of the recurrence relation is given by

$$a_n = c_1 x_1^n + c_2 x_2^n$$

For some constants c_1 and c_2, provided that $x_1 \neq x_2$ (Goodaire & Parmenter, 2006, p.171). Otherwise, the solution will be

$$a_n = c_1 x^n + c_2 n x^n$$

If $x_1 = x_2 = x$ (Goodaire & Parmenter, 2006, p.171). Below are some examples.

Example 3. Consider the recurrence relation

$$a_n = -5a_{n-1} + 6a_{n-2},$$

$n \geq 2$, Given that $a_0 = 5$, $a_1 = 8$ (Goodaire & Parmenter, 2006, p.174).

To solve this, we use the method of the characteristic equation for distinct roots.

$$
\begin{aligned}
&a_n = -5a_{n-1} + 6a_{n-2} \\
&a_n + 5a_{n-1} - 6a_{n-2} = 0 \\
&x_1 = -6, x_2 = 1 \\
&a_n = c_1\left(-6^n\right) + c_2\left(1^n\right) \\
&a_0 = 5 \rightarrow 5 = c_1\left(6^0\right) + c_2\left(1^0\right) \\
&5 = c_1 + c_2 \rightarrow equation 1 \\
&a_1 = 19 \rightarrow 19 = c_1\left(6^1\right) + c_2\left(1^1\right) \\
&19 = -6c_1 + c_2 \rightarrow equation 2
\end{aligned}
$$

Multiplying equation 1 by 6 and adding equation 1 to equation 2 we get:

$$
\begin{aligned}
&30 = 6c_1 + 6c_2 \leftarrow e1 \\
&19 = -6c_1 + c_2 \leftarrow e2 \\
&c_2 = 7, c_1 = -2 \\
&\therefore a_n = -2\left(-6^n\right) + 7\left(1^n\right)
\end{aligned}
$$

Where $n \geq 0$. A better explicit formula would be $a_n = -2\left(-6^n\right) + 7$ where $n \geq 0$. Notice that $\left(1^n\right)$ is not necessary since for $n \geq 0$, we have that $\left(1^n\right) = 1$

The next recurrence is an example of a second order linear homogeneous recurrence relation with repeated roots.

Example 4. Consider the recurrence relation

$$a_{n+1} = -8a_n - 16a_{n-1},$$

$n \geq 1$, Given that $a_0 = 5, a_1 = 17$ (Goodaire & Parmenter, 2006, p.175). To solve this, we use the method of the characteristic equation for non-distinct roots.

$$\begin{aligned}
&a_{n+1} = -8a_n - 16a_{n-1} \\
&a_{n+1} + 8a_n + 16a_{n-1} = 0 \\
&x^2 + 8x + 16 = 0 \rightarrow (x+4)^2 \\
&x = -4 \\
&a_n = c_1\left(-4^n\right) + c_2 n\left(-4^n\right) \\
&a_0 = 5 \rightarrow 5 = c_1\left(-4^0\right) + c_2 0\left(-4^0\right) \\
&5 = c_1 \rightarrow equation1 \\
&a_1 = 17 \rightarrow 17 = c_1\left(-4^1\right) + c_2 1\left(-4^1\right) \\
&17 = -4c_1 - 4c_2 \rightarrow equation2
\end{aligned}$$

Substituting equation 1 into equation 2 will result in:

$$\begin{aligned}
&5 = c_1 \rightarrow equation1 \\
&17 = -4c_1 - 4c_2 \rightarrow equation2 \\
&c_1 = 5, c_2 = -\frac{37}{4} \\
&\therefore a_n = 5(-4)^n - \frac{37}{4}(n)(-4)^n
\end{aligned}$$

Where $n \geq 0$.

In our final two examples, we will consider non-homogeneous second order linear recurrence relations, where $f(n) \neq 0$. Let p_n be any particular solution to the recurrence relation

$$a_n = ra_{n-1} + sa_{n-2} + f(n),$$

Ignoring initial conditions (Goodaire & Parmenter, 2006, p.172). Let q_n be the general solution to the associated homogeneous recurrence

$$a_n = ra_{n-1} + sa_{n-2},$$

Again ignoring initial conditions (Goodaire & Parmenter, 2006, p.172). Then it is easy to check

$$a_n = p_n + q_n$$

Is the general solution to the recurrence relation

$$a_n = ra_{n-1} + sa_{n-2} + f(n)$$

(Goodaire & Parmenter, 2006, p.172).

The initial conditions determine the constants in $a_n = p_n + q_n$ (Goodaire & Parmenter, 2006, p.172). We illustrate this method first with a simple example.

Example 5. Consider the recurrence relation

$$a_n = -2a_{n-1} + 15a_{n-2} + 24,$$

$n \geq 2$, Given that $a_0 = 1$, $a_1 = -1$ (Goodaire & Parmenter, 2006, p.174). To solve this, we use the method of the characteristic equation for non-homogeneous recurrences. We use p_n to denote a particular solution to the recurrence relation above. Below, I will solve for such a particular solution.

$$\begin{aligned}
&p_n = a + b_n \\
&p_n = -2[a + b(n-1)] + 15[a + b(n-2)] + 24 = \\
&-2(a + bn - b) + 15(a + bn - 2b) + 24 = \\
&-2a - 2bn + 2b + 15a + 15bn - 30b + 24 = \\
&a = 13a - 28b + 24 \rightarrow bn = 13bn \\
&13b = b \rightarrow 13b - b = 0 \rightarrow 12b = 0 \rightarrow b = 0 \\
&a = 13a - 28(0) + 24 \rightarrow -12a = 24 \rightarrow a = -2 \\
&p_n = a + bn \rightarrow -2 + (0)n \rightarrow p_n = -2
\end{aligned}$$

Next, we let q_n denote the general solution to the associated homogeneous recurrence relation. Below I will solve for q_n using the characteristic equation.

$$
\begin{aligned}
&a_n = -2a_{n-1} + 15a_{n-2} \\
&a_n + 2a_{n-1} - 15a_{n-2} = 0 \\
&x^2 + 2x - 15 = 0 \rightarrow (x+5)(x-3) \\
&x_1 = -5, x_2 = 3 \\
&q_n = c_1\left(-5^n\right) + c_2\left(3^n\right) \\
&a_n = p_n + q_n = -2 + c_1\left(-5^n\right) + c_2\left(3^n\right) \\
&a_0 = 1 \rightarrow 1 = -2 + c_1\left(-5^0\right) + c_2\left(3^0\right) \\
&3 = c_1 + c_2 \leftarrow equation 1 \\
&a_1 = -1 \rightarrow -1 = -2 + c_1\left(-5^1\right) + c_2\left(3^1\right) \\
&1 = -5c_1 + 3c_2 \leftarrow equation 2
\end{aligned}
$$

Multiplying equation 1 by 5 and adding to equation 2 results in:

$$
\begin{aligned}
&15 = 5c_1 + 5c_2 \leftarrow \text{ equation } 1 \\
&1 = -5c_1 + 3c_2 \leftarrow \text{ equation } 2 \\
&16 = 8c_2 \rightarrow c_2 = 2
\end{aligned}
$$

Next, substitute c_2 into the original equation:

$$
\begin{aligned}
&3 = c_1 + c_2 \\
&3 = c_1 + 2 \rightarrow c_1 = 1 \\
&\therefore a_n = \left(-5^n\right) + 2\left(3^n\right) - 2
\end{aligned}
$$

For all $n \geq 0$.

Finally, we solve a non-homogeneous second order linear recurrence relation in which $f(n) \neq 0$ or any other constant, but rather a variable.

Example 6. Consider the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 3n,$$

Where $n \geq 2$, $a_0 = 2, a_1 = 14$ (Goodaire & Parmenter, 2006, p.183). Here again, we let p_n denote the particular solution to the recurrence relation above. Below I will solve for the particular solution.

$$
\begin{aligned}
&p_n = a + bn \\
&p_n = 5\big(a + b(n-1)\big) - 6\big(a + b(n-2)\big) + 3n \\
&p_n = 5a + 5bn - 5b - 6a - 6bn + 12b + 3n \\
&bn = 5bn - 6bn + 3n \\
&bn = -bn + 3n \\
&bn = n(-b + 3) \\
&b = -b + 3 \rightarrow 2b = 3 \rightarrow b = \frac{3}{2} \\
&a = 5a - 6a + 12b - 5b \\
&a = -a + 7b \\
&2a = 7b
\end{aligned}
$$

$$2a = 7\left(\frac{3}{2}\right) \rightarrow a = \frac{21}{4}$$

$$p_n = a + bn \rightarrow \therefore p_n = \frac{21}{4} + \frac{3}{2}n$$

Now that we have the particular solution, we find the general solution to the associated homogeneous recurrence, and combine it with the above.

$$
\begin{aligned}
&a_n = 5a_{n-1} + 6a_{n-2} \\
&a_n - 5a_{n-1} - 6a_{n-2} = 0 \\
&x^2 - 5x + 6 \\
&(x-2)(x-3) = 0 \rightarrow x_1 = 2, x_2 = 3 \\
&q_n = c_1(2^n) + c_2(3^n) \\
&a_n = q_n + p_n \\
&a_n = \frac{21}{4} + \frac{3}{2}n + c_1(2^n) + c_2(3^n) \\
&a_0 = 2 \rightarrow 2 = \frac{21}{4} + \frac{3}{2}(0) + c_1(2^0) + c_2(3^0) \\
&2 = \frac{21}{4} + c_1 + c_2 \rightarrow -\frac{13}{4} = c_1 + c_2 \leftarrow equation 1 \\
&a_1 = 14 \rightarrow 14 = \frac{21}{4} + \frac{3}{2}(1) + c_1(2^1) + c_2(3^1) \\
&\frac{29}{4} = 2c_1 + 3c_2 \leftarrow equation 2
\end{aligned}
$$

Multiplying equation 1 by -2 and adding it to equation 2 results in the following:

$$
\begin{aligned}
&\frac{26}{4} = -2c_1 - 2c_2 \leftarrow equation 1 \\
&\frac{29}{4} = 2c_1 + 3c_2 \leftarrow equation 2 \\
&\frac{55}{4} = c_2
\end{aligned}
$$

Substituting c_2 back into the original equation 1 results in the following:

$$
\begin{aligned}
&-\frac{13}{4} = c_1 + \frac{55}{4} \rightarrow c_1 = -17 \\
&a_n = p_n + q_n \\
&\therefore a_n = \frac{21}{4} + \frac{3}{2}n - 17(2^n) + \frac{55}{4}(3^n)
\end{aligned}
$$

For all $n \geq 0$

2.3: Generating Functions

The method of generating functions is another technique to solve recurrence relations. A *generating function* for a sequence of numbers is a polynomial or a power series that has the terms of the sequence as its coefficients (Goodaire & Parmenter, 2006, p.176). A power series is simply treated like a polynomial that goes on infinitely. A finite sequence will have a generating function with an expression that is in the form

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$$

(Goodaire & Parmenter, 2006, p.176).

On the other hand, a sequence with infinitely many nonzero terms will have a generating function that also has infinitely many nonzero terms. There is an obvious correspondence between generating functions and sequences; that is,

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \leftrightarrow a_0, a_1, a_2a_3 \ldots$$

In other words, the generating function of a sequence $a_0, a_1, a_2, \ldots$ is the expression

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots$$

(Goodaire & Parmenter, 2006, p.176).

Generating functions can often be expressed as the quotient of polynomials. An important example is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots,$$

Thus $\frac{1}{1-x}$ is the generating function for the sequence $1,1,1\ldots$ (Goodaire & Parmenter, 2006, p.177). A variation of this gives another important example: if $a \in R$ and we replace x with ax in the above, then we get the following result:

$$\frac{1}{1-ax} = 1 + (ax) + (ax)^2 + (ax)^3 + \ldots = 1 + ax + a_2x^2 + a_3x^3 + \ldots,$$

So $\frac{1}{1-ax}$ is the generating function for the sequence

$$1, a, a_1, a_2, a_3, \ldots,$$

Notice that this is the geometric sequence with first term 1 and common ratio a (Goodaire & Parmenter, 2006, p.177,180). Below are some examples of solving recurrence relations using generating functions.

Example 7. Consider the recurrence relation

$$a_n = 2a_{n-1}, n \geq 1, a_0 = 1$$

(Goodaire & Parmenter, 2006, p.181).

To solve this, we let *f(x)* be the generating function for the solution sequence and we compute as follows:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$f(x) = a^0 x^0 + \sum_{n=1}^{\infty} (2a_{n-1}) x^n$$

$$f(x) = 1 + 2\sum_{n=1}^{\infty} (a_{n-1}) x^n$$

$$f(x) = 1 + 2x\sum_{n=0}^{\infty} (a_{n-1}) x^{n-1}$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow f(x) = 1 + 2xf(x)$$

$$f(x) - 2xf(x) = 1 \rightarrow f(x)(1-2x) = 1$$

$$f(x) = \frac{1}{1-2x} \rightarrow f(x) = \sum_{n=0}^{\infty} (2x)^n \rightarrow f(x) = \sum_{n=0}^{\infty} 2^n x^n$$

$$\therefore a_n = 2^n$$

For all $n \geq 0$.

Example 8. Consider the recurrence relation

$$a_n = 3a_{n-1} + 1,$$

For $n \geq 1$, given $a_0 = 1$ (Goodaire & Parmenter, 2006, p.181).

To solve this, we let *f(x)* be the generating function for the solution sequence and we compute as follows:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n$$

$$1 + \sum_{n=1}^{\infty} (3a_{n-1} + 1) x^n \rightarrow 1 + \sum_{n=1}^{\infty} 3a_{n-1} x^n + \sum_{n=1}^{\infty} x^n$$

Note that $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1$, so we have that

$$1+3x\sum_{n=1}^{\infty}a_{n-1}x^{n-1}+\left(\frac{1}{1+x}-1\right)$$

$$f(x)=1+3xf(x)+\frac{1}{1-x}-1$$

$$f(x)-3xf(x)=\frac{1}{1-x}$$

$$f(x)(1-3x)=\frac{1}{1-x}\rightarrow f(x)=\frac{1}{(1-x)(1-3x)}$$

$$\frac{A}{1-x}+\frac{B}{1-3x}=\frac{A(1-3x)+B(1-x)}{(1-x)(1-3x)}$$

$$\frac{A-A3x+B-Bx}{(1-x)(1-3x)}=f(x)$$

$$A-A3x+B-Bx=1+0x$$

$$A+B+(-3A-B)x=1+0x$$

$$A+B=1\leftarrow equation1$$

$$-3A-B=0\leftarrow equation2$$

$$-2A=1\rightarrow A=-\frac{1}{2},B=\frac{3}{2}$$

$$f(x)=\frac{-1/2}{1-x}+\frac{3/2}{1-3x}\rightarrow\left(-\frac{1}{2}\right)\left(\frac{1}{1-x}\right)+\left(\frac{3}{2}\right)\left(\frac{1}{1-3x}\right)$$

$$f(x)=-\frac{1}{2}\sum_{n=0}^{\infty}x^n+\frac{3}{2}\sum_{n=0}^{\infty}(3x)^n$$

$$\therefore a_n=-\frac{1}{2}+\frac{3}{2}(3^n)$$

This next problem is an example of a recurrence relation that is not second order and could be solved using the characteristic polynomial, but will be solved here using generating functions.

Example 8. Consider the third-order recurrence relation

$$a_n=a_{n-1}+a_{n-2}-a_{n-3},$$

For $n\geq 3$ given $a_0=2, a_1=-1, a_2=3$ (Goodaire & Parmenter, 2006, p.182).

To solve this, we let *f(x)* be the generating function for the solution sequence and we compute as follows:

$$
\begin{aligned}
&f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n \\
&xf(x) = a_0 x + a_1 x^2 + a_2 x^3 + \ldots + a_{n-1} x^n \\
&x^2 f(x) = a_0 x^2 + a_1 x^3 + \ldots + a_{n-2} x^n \\
&x^3 f(x) = a_0 x^3 + \ldots a_{n-3} x^n \\
&a_n - a_{n-1} - a_{n-2} + a_{n-3} = 0 \\
&f(x) - xf(x) - x^2 f(x) + x^3 f(x) = \\
&a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1 - a_0)x^3 \\
&2 + (-1-2)x + (3-(-1)-2)x^2 + (0-3-(-1)+2)x^3 \rightarrow 2 - 3x + 2x^2 \\
&f(x) - xf(x) - x^2 f(x) + x^3 f(x) = 2 - 3x + 2x^2 \\
&f(x)(1 - x - x^2 + x^3) = 2 - 3x + 2x^2 \rightarrow f(x) = \frac{2 - 3x + 2x^2}{(1 - x - x^2 + x^3)} \\
&f(x) = \frac{2 - 3x + 2x^2}{(1+x)(1-x)^2} \rightarrow \frac{Ax + B}{(1-x)^2} + \frac{C}{(1+x)} = \frac{2 - 3x + 2x^2}{(1+x)(1-x)^2} \\
&Ax + B(1+x) + C(1-x)^2 = 2 - 3x + 2x^2 \\
&Ax + Ax^2 + B + Bx + C - 2Cx + Cx^2 = 2 - 3x + 2x^2 \\
&\begin{bmatrix} B + C = 2 \\ Ax + Bx - 2Cx = -3 \\ Ax^2 + Cx^2 = 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & -2 & -3 \\ 1 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 7/4 \end{bmatrix}
\end{aligned}
$$

$$f(x)=\frac{1/4\,x+1/4}{(1-x)^2}\frac{7/4}{(1+x)}$$

$$f(x)=\left[\frac{1}{4}x+\frac{1}{4}\right]\left(\frac{1}{(1-x)^2}\right)+\frac{7}{4}\left(\frac{1}{1+x}\right)$$

$$\frac{1}{(1-x)^2}=\frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{d}{dx}(1-x)^{-1}=-1(1-x)^2(-1)=\frac{1}{(1-x)^2}$$

$$\left(\frac{1}{1-x}\right)=1+x+x^2+x^3+x+\ldots+x^n$$

$$\left(\frac{1}{(1-x)^2}\right)=\frac{d}{dx}\left(\frac{1}{1-x}\right)=1+2x+3x^2+4x^3+\ldots+(n+1)x^n$$

$$\frac{1}{4}x\left[\sum_{n=0}^{\infty}(n+1)x^n\right]+\frac{1}{4}+\frac{7}{4}\sum_{n=0}^{\infty}(-1)^n x^n$$

$$\frac{1}{4}x\left[\sum_{n=0}^{\infty}(n+1)x^n\right]+\frac{1}{4}\left[(n+1)+\frac{7}{4}(-1)^n\right]x^n$$

$$\therefore a_n=\frac{1}{4}(2n+1)+\frac{7}{4}(-1)^n,\forall n\geq 0$$

Example 9. Consider the recurrence relation

$$a_n=4a_{n-1}+5a_{n-2}+3^n$$

For $n\geq 2$ given $a_0=4, a_1=-1$ (Goodaire & Parmenter, 2006, p.183). To solve this, we let *f(x)* be the generating function for the solution sequence and compute:

$$f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n$$
$$4xf(x)=4a_0x+4a_1x^2+\ldots+a_{n-1}x^n$$
$$5x^2f(x)=5a_0x^2+\ldots+a_{n-2}x^n$$
$$3^n=\left(3^0x^0+3x+3^2x^2+3^nx^n\right)=\frac{1}{1-3x}$$
$$a_n-4a_{n-1}-5a_{n-2}-3^n=0$$
$$f(x)-4xf(x)-5x^2f(x)-\frac{1}{1-3x}=$$

$$a_0 - 3^0 + (a_1 - 4a_0 - 3)x + (a_2 - 4a_1 - 5a_0 - 3^2)x^2 =$$
$$4 - 1 + (-1 - 4(4) - 3)x + (25 - 4(-1) - 5(4) - 9)x^2 =$$
$$3 + (-20x)$$
$$f(x)(1 - 4x - 5x^2) = \frac{1}{1-3x} + 3 - 20x$$
$$\frac{1 + (1-3x)(3-20x)}{1-3x} = \frac{4 - 29x + 60x^2}{1-3x}$$
$$f(x)(1 - 4x - 5x^2) = \frac{4 - 29x + 60x^2}{1-3x}$$
$$f(x)\frac{4 - 29x + 60x^2}{(1-3x)(1-4x-5x^2)}$$
$$f(x) = \frac{4 - 29x + 60x^2}{(1-3x)(1-5x)(1+x)}$$
$$f(x) = \frac{4 - 29x + 60x^2}{(1-3x)(1-5x)(1+x)} = \frac{A}{1-3x} + \frac{B}{1-5x} + \frac{C}{1+x}$$
$$4 - 29x + 60x^2 = A(1-5x)(1+x) + B(1-3x)(1+x) + C(1-3x)(1-5x)$$
$$= A(1 - 4x - 5x^2) + B(1 - 2x - 3x^2) + C(1 - 8x + 15x^2)$$
$$A + B + C = 4$$
$$-4A - 2B - 8C = -29$$
$$-5A - 3B + 15C = 60$$
$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ -4 & -2 & -8 & -29 \\ -5 & -3 & 15 & 60 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1.125 \\ 0 & 1 & 0 & 1.25 \\ 0 & 0 & 1 & 3.875 \end{bmatrix} = \begin{matrix} A = -9/8 \\ B = 5/4 \\ C = 31/8 \end{matrix}$$
$$F(x) = \frac{-9/8}{1-3x} + \frac{5/4}{1-5x} + \frac{31/81}{1+x}$$
$$\frac{-9}{8}\left(\frac{1}{1-3x}\right) + \frac{5}{4}\left(\frac{1}{1-5x}\right) + \frac{31}{8}\left(\frac{1}{1+x}\right) =$$
$$\frac{-9}{8}\sum_{n=0}^{\infty} 3^n x^n + \frac{5}{4}\sum_{n=0}^{\infty} 5^n x^n + \frac{31}{8}\sum_{n=0}^{\infty} 1^n x^n =$$
$$\sum_{n=0}^{\infty}\left[\frac{-9}{8}(3^n) + \frac{5}{4}(5^n) + \frac{31}{8}\right]x^n$$
$$\therefore a_n = \frac{-9}{8}(3^n) + \frac{5}{4}(5^n) + \frac{31}{8}(-1)^n, \forall n \geq 0$$

When n is odd then it is -1 and when n is even then it is 1.

2.3: Linear Algebra

The term *characteristic polynomial,* as we have been using it, has its origins in linear algebra (Goodaire & Parmenter, 2006, p.171). The term characteristic polynomial is connected to its use in the study of Eigen values (Goodaire & Parmenter, 2006, p.174). Specifically, the recurrence relation

$$a_n = ra_{n-1} + sa_{n-2},$$

Where r and s are constants, can be expressed in matrix form

$$\begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix},$$

Which is the same as saying

$$v_n = Av_{n-1},$$

Where $v_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$ and $A = \begin{bmatrix} r & s \\ 1 & 0 \end{bmatrix}$.

(Goodaire & Parmenter, 2006, p.174).

The characteristic polynomial of this matrix A is

$$\det \begin{bmatrix} r-x & s \\ 1 & -x \end{bmatrix} = x^2 - rx - s,$$

Which is the same as what we have been calling the characteristic polynomial of the recurrence relation (Goodaire & Parmenter, 2006, p.174). Below are some examples of the use of linear algebra and the characteristic polynomial to solve recurrence relations.

Example 10. Consider the recurrence relation

$$a_{n+1} = 3a_n - 2a_{n-1}$$

For $n \geq 1$ given $a_0 = -4, a_1 = 0$ (Goodaire, 2003, p.468). To solve this, we formulate it as a matrix condition:

$$\begin{aligned} &v_n = A_n \bullet v_{n-1} \\ &\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} \\ &A^n v_0 = A^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = A^n \begin{bmatrix} 0 \\ -4 \end{bmatrix} \end{aligned}$$

Next, we will get the characteristic polynomial of A by the diagonalization of A.

$$\begin{aligned} &(A - \lambda I) = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix} \\ &\det(A - \lambda I) = (3-\lambda)(-\lambda) - (-2)(1) \\ &\lambda^2 - 3\lambda + 2 = 0 \to (\lambda - 2)(\lambda - 1) \to \lambda_1 = 1, \lambda_2 = 2 \\ &D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

The numbers λ_1 and λ_2 are the Eigen values of A and $\begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix}$ is the diagonal form of A. To find the Eigen space for $\lambda_1 = 1$ we have:

$$\begin{aligned} &(A - \lambda I)x = 0 \\ &\begin{bmatrix} 3-\lambda_1 & -2 \\ 1 & -\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \to \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \\ &2x_1 - 2x_2 = 0 \\ &x_1 - x_2 = 0 \end{aligned}$$

Where, $x_2 = t_1$ is free and $x_1 = x_2 = t_1$. This gives us $x = \begin{bmatrix} t_1 \\ t_1 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To find the Eigen space for $\lambda_2 = 2$ we have:

$$(A-\lambda I)x=0$$
$$\begin{bmatrix} 3-\lambda_2 & -2 \\ 1 & -\lambda_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0 \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$
$$x_1-2x_2=0$$
$$x_1-2x_2=0$$

Where, $x_2 = t_2$ is free and $x_1 = 2x_2 = 2t_2$. This gives us $x=\begin{bmatrix} 2t_2 \\ t_2 \end{bmatrix}=t_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Next, we will write the matrices P, P^{-1}, D to solve for A.

$$P=\begin{vmatrix} t_1 & t_2 \\ t_1 & t_2 \end{vmatrix}=\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$
$$D=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The matrix P^{-1} is obtained by observing that if $P=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$ then P^{-1} is

$$P^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\rightarrow\frac{1}{(1)(1)-(2)(1)}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$$
$$P^{-1}=-\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\rightarrow\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$
$$P^{-1}AP=D\Rightarrow A=PDP^{-1}$$
$$v_n=A^n v_0=\left(PDP^{-1}\right)v_0=\left(PDP^{-1}\right)\begin{bmatrix} 0 \\ -4 \end{bmatrix}$$
$$P^{-1}v_0=\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -4 \end{bmatrix}=\begin{bmatrix} -8 \\ 4 \end{bmatrix}$$
$$PD^n=\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix}=\begin{bmatrix} 1 & 2^{n+1} \\ 1 & 2^n \end{bmatrix}$$
$$v_n=PD^n \bullet P^{-1}v_0$$
$$v_n=\begin{bmatrix} 1 & 2^{n+1} \\ 1 & 2^n \end{bmatrix}\begin{bmatrix} -8 \\ 4 \end{bmatrix}=\begin{bmatrix} -8+4\left(2^{n+1}\right) \\ -8+4\left(2^n\right) \end{bmatrix}\begin{matrix} a_{n+1} \\ a_n \end{matrix}$$
$$\therefore a_n=-8+4\left(2^n\right)$$

A better explicit formula is $a_n=4\left[-2+\left(2^n\right)\right]$ for all $n\geq 0$.

Example 11. Consider the recurrence relation

$$a_{n+1} = a_n + 12a_{n-1}$$

For $n \geq 1$ given $a_1 = -8, a_2 = 1$ (Goodaire, 2003, p.468). To solve this, we first compute the natural initial condition:

$$a_2 = a_1 + 12a_0 \rightarrow 1 = -8 + 12a_0 \rightarrow 9 = 12a_0$$
$$\frac{9}{12} = a_0 \rightarrow \frac{3}{4} = a_0$$

Next we formulate the problem in terms of matrices:

$$v_n = A_n \bullet v_{n-1}$$
$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$
$$A^n v_0 = A^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = A^n \begin{bmatrix} -8 \\ 3/4 \end{bmatrix}$$

Next, we will get the characteristic polynomial of A by the diagonalization of A.

$$(A - \lambda I) = \begin{bmatrix} 1 & 12 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 12 \\ 1 & -\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = (1-\lambda)(-\lambda) - (12)(1)$$
$$\lambda^2 - \lambda - 12 = 0 \rightarrow (\lambda - 4)(\lambda + 3) \rightarrow \lambda_1 = -3, \lambda_2 = 4$$
$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$$

The numbers λ_1 and λ_2 are Eigenvalues of A and $\begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix}$ is the diagonal form of A.

To find the Eigen space for $\lambda_1 = -3$ we have:

$$(A-\lambda I)x=0$$
$$\begin{bmatrix}1-\lambda_1 & 12\\ 1 & -\lambda_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0\rightarrow\begin{bmatrix}4 & 12\\ 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$
$$4x_1+12x_2=0$$
$$x_1+3x_2=0$$

Where, $x_2=t_1$ is free and $x_1=-3x_2=-3t_1$. This gives us $x=\begin{bmatrix}-3t_1\\ t_1\end{bmatrix}=t_1\begin{bmatrix}-3\\ 1\end{bmatrix}$.

To find the Eigen space for $\lambda_2=4$ we have:

$$(A-\lambda I)x=0$$
$$\begin{bmatrix}1-\lambda_2 & 12\\ 1 & -\lambda_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0\rightarrow\begin{bmatrix}-3 & 12\\ 1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$
$$-3x_1+12x_2=0$$
$$x_1-4x_2=0$$

Where, $x_2=t_2$ is free and $x_1=4x_2=4t_2$. This gives us $x=\begin{bmatrix}4t_2\\ t_2\end{bmatrix}=t_2\begin{bmatrix}4\\ 1\end{bmatrix}$.

Next, we will write the matrices P,P^{-1},D to solve for A.

$$P=\begin{bmatrix}t_1 & t_2\\ t_1 & t_2\end{bmatrix}=\begin{bmatrix}-3 & 4\\ 1 & 1\end{bmatrix}$$
$$D=\begin{bmatrix}-3 & 0\\ 0 & 4\end{bmatrix}$$

The matrix P^{-1} is obtained by noticing that if $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & 1 \end{bmatrix}$ then P^{-1} is

$$P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \rightarrow \frac{1}{(-3)(1)-(4)(1)}\begin{bmatrix} 1 & -4 \\ -1 & -3 \end{bmatrix}$$

$$P^{-1} = -\frac{1}{7}\begin{bmatrix} 1 & -4 \\ -1 & -3 \end{bmatrix} \rightarrow \frac{1}{7}\begin{bmatrix} -1 & 4 \\ 1 & 3 \end{bmatrix}$$

$$P^{-1}AP = D \Rightarrow A = PDP^{-1}$$

$$v_n = A^n v_0 = \left(PDP^{-1}\right)v_0 = \left(PDP^{-1}\right)\begin{bmatrix} -8 \\ 3/4 \end{bmatrix}$$

$$P^{-1}v_0 = \frac{1}{7}\begin{bmatrix} -1 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -8 \\ 3/4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 11 \\ -23/4 \end{bmatrix}$$

$$PD^n = \begin{bmatrix} -3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3^n & 0 \\ 0 & 4^n \end{bmatrix} = \begin{bmatrix} -3^{n+1} & 4^{n+1} \\ -3^n & 4^n \end{bmatrix}$$

$$v_n = PD^n \bullet P^{-1}v_0$$

$$v_n = \frac{1}{7}\begin{bmatrix} -3^{n+1} & 4^{n+1} \\ -3^n & 4^n \end{bmatrix}\begin{bmatrix} 11 \\ -23/4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 11\left(-3^{n+1}\right) + -\frac{23}{4}\left(4^{n+1}\right) \\ 11\left(-3^n\right) + -\frac{23}{4}\left(4^n\right) \end{bmatrix}\begin{matrix} a_{n+1} \\ a_n \end{matrix}$$

$$\therefore a_n = \frac{1}{7}\left[11\left(-3^n\right) - \frac{23}{4}\left(4^n\right)\right]$$

For all $n \geq 0$.

Chapter 3

John Pell: An "obscure" English Mathematician

(Webster, 2006)

John Pell (1611-1685) has been considered the most enigmatic of the seventeenth-century mathematicians (Steadall, 2002, p.126). He is a significant figure in the intellectual history of 17th century Europe not because of his published work but more because of his activities, contacts and correspondence (Malcolm, 2000, p.2). He was well read in classical and contemporary mathematics, but does not have much of a following or reputation among mathematicians because he hardly published anything (Steadall, 2002, p.126). From the few books and papers that he did publish, the one that is best known is an "Introduction to Algebra," published in 1668 (Steadall, 2002, p.126). During his adult life, Pell had correspondence with Descartes, Leibniz, Cavendish, Mersenne, Hartlib, Collins and others (Steadall, 2002, p.127).

One of Pell's main mathematical focuses was making and studying mathematical tables (Steadall, 2002, p.127). He particularly liked tables of squares, sums of squares, primes and composites, constant differences, logarithms, antilogarithms, trigonometric functions, as well as many others (Steadall, 2002, p.127-128). Pell liked to be anonymous, as is evident by the many booklets that he created of his tables (Steadall, 2002, p.128). The tables all had a title page, but he did not list himself as the author (Steadall, 2002, p.128). Only one of his tables was ever published in 1672, which was a table of the first 10,000 square numbers

(Steadall, 2002, p.148). Pell's fascination with tables continued with tables of primes and composites. The first fair-sized tables of primes and composites, giving the least prime factors not divisible by 2 or 5, was published by J.H. Rahn as an appendix to his *Teusche Algebra* (1659) (Burton, 2011, p.507). Pell extended this table to include numbers up to 100,000 (Burton, 2011, p.507). In regards to tables, Pell also worked with Walter Warner, who was the last surviving member of Thomas Hariot's inner circle (Malcolm, 2000). Pell had an ambitious project, which they worked on together, which was the construction of tables of antilogarithms (Steadall, 2002, p.131). Warner died in 1643 before the calculations where finished and Pell went on to other things (Steadall, 2002, p.132).

In Pell's best-known published work, "An Introduction to Algebra", he explains the rules and notation for handling and simplifying equations (Steadall, 2002, p.136). After that, the remaining and greater part of the book is devoted to "The Resolution of divers Arithmetical and Geometrical Problemes" (Steadall, 2002, p.137). It is in this book that the division symbol appears $\div$ (Steadall, 2002, p.137). In another work, "Idea of Mathematics", it presents evidence confirming Pell's responsibility for the introduction of the division sign $\div$ (Malcolm, 2000, p.2). During his three years of teaching in Amsterdam (1643-1646) much of his time and energy went into refuting "A Quaduture of the Circle" published by the Danish mathematician Longomontanus (Steadall, 2002, p.132). His efforts led him to the discovery of the double-angle tangent formula: $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ (Steadall, 2002, p.132).

John Pell, at least in name, is probably best known for the Pell Sequence and the Pell Equation. Although the Pell Sequence is named after John Pell, I could not find any other good definitive published material that he actually studied and contributed to this sequence extensively (Gullberg, 1997, p.288). He is best known though, probably for the Pell Equation, which is mistakenly named after him by Euler (Sandifer, 2007, p.63). Pell lived about one hundred years earlier than Euler (Sandifer, 2007, p.63). Much of elementary number theory has its roots in Euler who cites as his inspiration the works of Fermat, Diophantus, Goldbach and Pell among others (Sandifer, 2007, p.63). As James Tattersall states, "Euler after a cursory reading of Wallis's *Opera Mathematica* mistakenly attributed the first serious study of non-trivial solutions to equations of the form $x^2 - dy^2 = 1$ where $x \neq y$ and $y \neq 0$ to John Pell, mathematician to Oliver Cromwell." (Tattersall, 2005, p.274). I found numerous books about the Pell Equation that state that Euler mistakenly named the equation after Pell.

John Pell was married and had eight children. He had constant financial trouble throughout his life and was twice imprisoned for unpaid debts (Malcolm, 2000). He taught at the Gymnasium in Amsterdam (1643-1646) and from 1654-1658 he was Cromwell's envoy to Switzerland.

In the end as I have studied John Pell I come away with the impression of a man who loved reading, studying, working on projects, teaching, corresponding and working with mathematics. He is not well known in the history of math because of his desire to be anonymous and for his lack of publishing with significant importance and with very little material published. He seems to have a reputation in life as well as in death as a mathematician who was easily distracted, had multiple projects going on at once and had many unfinished projects. He had correspondence and was held with high regard, as stated earlier, by his contemporaries in England as well as Continental Europe. Within this group though, he was a minor figure with not much notoriety. He loved mathematical tables and table making. In summation, he is best known for the Pell Sequence, Pell Equation, the division sign and the double-angle tangent formula, of which the first two he did not have a significant role in contributing to the study of the topic.

Because Pell liked to remain anonymous he did not publish as much as his contemporaries and to this day is one of the more "obscure" mathematicians (Webster, 2006). But, in spite of everything, he dedicated a large part of his life to mathematics and for that he is recognized.

Chapter 4

The Pell Sequence, its history and some amazing properties

4.1 The Pell Sequence: History and Properties

The *Pell Sequence*, also known as the sequence of *Pell Numbers*, is found in the **Encyclopedia of Integer Sequences** under the code of M1413 (Sloane & Plouffe, 1995). It is defined by the recurrence relation of,

$$p_n = 2p_{n-1} + p_{n-2}$$

With the initial conditions of $p_0 = 1$ and $p_1 = 2$ (Goodaire & Parmenter, 2006, p.182) This generates the sequence of

$$1, 2, 5, 12, 29, 70, 169, 408, \ldots$$

With a closed-form formula of

$$p_n = \frac{2+\sqrt{2}}{4}\left(1+\sqrt{2}\right)^n + \frac{2-\sqrt{2}}{4}\left(1-\sqrt{2}\right)^n$$

For all $n \geq 0$. In the next section, we will deduce this expression using our four techniques for solving recurrences. In this chapter, however, our goal is simply to give the reader a tour of some fascinating properties of this sequence, primarily without proof. Before we begin, however, we remark that sometimes the initial conditions are defined as $p_0 = 0$ and $p_1 = 1$. A shorter version of the previous formula is

$$p_n \approx \frac{2+\sqrt{2}}{4}\left(1+\sqrt{2}\right)^n$$

For all $n \geq 0$ (Goodaire & Parmenter, 2006, p.182). This formula is not exact, but it will give us a value closest to the nearest integer in the sequence. The Pell Sequence is named after the English mathematician of the seventeenth century, John Pell (Gullberg, 1997, p.288).

The Fibonacci Sequence, when represented geometrically, is closely related to a rectangle known as the "Golden Rectangle" (Bicknell, 1975, p.345). The equation that defines the Pell Numbers, when represented geometrically, can be similarly associated with a rectangle, which is often referred to as the "Silver Rectangle." In this rectangle, the ratio of length to width is denoted by length y and width 1 (Bicknell, 1975, p.345). When the two squares with the side equal to the width are taken out of the rectangle, the rectangle that remains has the same ratio of length to width as did the original rectangle (Bicknell, 1975, p.345).

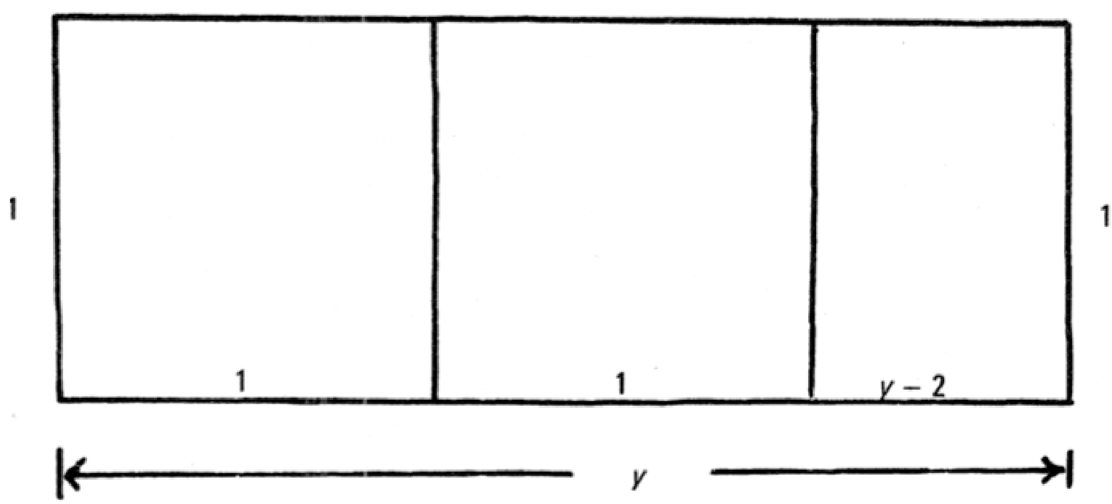

Figure 1

(Bicknell, 1975, p.346, image)

Algebraically, this ratio can be represented by the proportion

$$\frac{y}{1} = \frac{1}{y-2}.$$

Solving this leads to the consideration of the quadratic

$$y^2 - 2y - 1 = 0,$$

Which leads to the dominant root $y = \left(\frac{2+\sqrt{8}}{2}\right)$

(Bicknell, 1975, p.345).

Some interesting facts about the Pell Numbers can be deduced from the explicit formula. For example:

- The only triangular Pell Number is 1 (McDaniel, 1996).
- The largest proven prime has an index of $13,339$ with $5,106$ digits (Weisstein, 2011).
- The largest known probable prime has an index of $90,197$ with $34,525$ digits (Weisstein, 2011).
- For a Pell Number P_n to be prime, n has to be prime as well (Weisstein, 1999-2011).
- The only Pell Numbers that are squares, cubes or any other power higher are $0,1,169$ (Cohn & Pethos, 1992, 1996).
- The Pell Numbers or Pell Sequence $1,2,5,12,29,70,169,408,\ldots$ are the denominators of fractions that are the closest rational approximations to the square root of 2 (Flannery, 2006).

This last point may require a bit of illustration. For example, the digits of the square root of two are $\sqrt{2} = 1.414214\ \ldots$, but the method of continued fractions gives the following sequence of rational approximations:

$$\frac{1}{1},\frac{3}{2},\frac{7}{5},\frac{17}{12},\frac{41}{29},\frac{99}{70},\ldots$$

Which continually get closer to $\sqrt{2}$ (Flannery, 2006). There are many patterns in this fractional sequence, but the most obvious here is that the denominators are the Pell numbers! Below we list some other intriguing patterns in this sequence of fractions that are related to the Pell sequence.

- The sum of the numerator and denominator of the previous term is the denominator of the current term. For example, $3+2=5$ which includes the fractions in the sequence $\frac{3}{2}, \frac{7}{5}$ (Flannery, 2006).
- The numerator of the current fraction is the sum of the numerator and two times the denominator of the previous fraction. For example, if we take the two fractions in the sequence, $\frac{7}{5}, \frac{17}{12}$ then $7 + (2 \bullet 5) = 17$. Symbolically this can be defined as $\frac{m}{n} \rightarrow \frac{m+2n}{m+n}$ where $\frac{m}{n}$ is the current term and the next term will be $\frac{m+2n}{m+n}$ (Flannery, 2006, p.118).
- Since each fraction in the sequence is represented by $\frac{m}{n}$ then each fraction of the sequence will either follow $m^2 = 2n^2 - 1 \rightarrow m^2 - 2n^2 = -1$ or $m^2 = 2n^2 + 1 \rightarrow m^2 - 2n^2 = 1$ (Flannery, 2006, p.80).

For example, $\frac{1}{1} \rightarrow 1^2 - 2(1)^2 = -1, \frac{3}{2} \rightarrow 3^2 - 2(2)^2 = 1$, etc. with each fraction thereafter alternating between 1 and -1 (Flannery, 2006, p.80). To see this, note that

$$(m+2n)^2 - (m+n)^2 \rightarrow \left(m^2 + 4mn + 4n^2\right) - 2\left(m^2 + 2mn + n^2\right) \rightarrow m^2 + 2n^2$$

So either $-m^2 - 2n^2 = -\left(m^2 - 2n^2\right)$ produces a result of 1 or -1 (Flannery, 2006, p.82). Below is m^2 isolated on the right side of the equation (Flannery, 2006, p.21).

$$1^2 = 2(1)^2 - 1$$
$$1^2 = 2(2)^2 + 1$$
$$1^2 = 2(5)^2 - 1$$
$$1^2 = 2(12)^2 + 1$$
$$1^2 = 2(29)^2 - 1$$
$$1^2 = 2(70)^2 + 1$$

Then, if we divide both sides by n^2 or the numbers in the Pell Sequence we get

$$\left(\frac{1}{1}\right)^2 = 2 - \frac{1}{1^2}$$
$$\left(\frac{3}{2}\right)^2 = 2 - \frac{1}{2^2}$$
$$\left(\frac{7}{5}\right)^2 = 2 + \frac{1}{5^2}$$
$$\left(\frac{17}{12}\right)^2 = 2 - \frac{1}{12^2}$$
$$\left(\frac{41}{29}\right)^2 = 2 + \frac{1}{29^2}$$
$$\left(\frac{99}{70}\right)^2 = 2 - \frac{1}{70^2}$$

(Flannery. 2006, p.22)

These alternating fractions will either be above or below the $\sqrt{2}$. The fractions $\frac{1}{1}, \frac{7}{5}, \frac{41}{29}$, are below the $\sqrt{2}$ and $\frac{3}{2}, \frac{17}{12}, \frac{99}{70}$, are above the $\sqrt{2}$. Each successive fraction is a better approximation to the $\sqrt{2}$ (Flannery, 2006, p.24-25).

- There is a recurrence relation that will also work for the numerators as well in the sequence, $p_n = 2p_{n-1} + p_{n-2}$ where $p_0 = 1, p_1 = 3$. The numerators in the sequence are $1, 3, 7, 17, 41, 99, 239, 577, \ldots$ each successive term in the Pell sequence $1, 2, 5, 12, 29, 70, 169, 408, \ldots$ is the denominator in the fractional sequence.

This can produce the numerator of the fractional sequence to the nearest integer if you multiply each number in the denominator by $\sqrt{2}$ (Flannery, 2006, p.191). For example, $\frac{1}{1}$ will be $\sqrt{2} = 1.41421 \ldots$ which is approximately 1, $\frac{3}{2}$ will be $2\sqrt{2} = 2.82843 \ldots$ which is approximately 3 and $\frac{7}{5}$ will be $5\sqrt{2} \approx 7.071 \ldots$ which is approximately 7 (Flannery, 2006, p.191).

- Starting with $\frac{1}{1}$, we can generate each successive term in the sequence if we take $\frac{m}{n} \to \frac{m+2n}{m+n}$ and if we use the rule $\frac{m}{n} \to \frac{3m+4n}{2m+3n}$ beginning with $\frac{1}{1}$ this will give us the fraction immediately after $\frac{m+2n}{m+n}$. (Flannery, 2006, p.117).

Some identities that might be of interest include the generating function

$$\frac{1}{1-2x-x^2} = \sum_{i=1}^{\infty} P_n x^n$$

And

$$p_n^2 = \frac{P_n P_{n+1}}{2}.$$

(Bicknell, 1975, p.346).

A matrix can also generate the Pell Numbers,

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

And

$$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}.$$

(Bicknell, 1975, p.347).

An identity from the matrix is,

$$\det M^n = (\det M)^n = (-1)^n = P_{n+1}P_{p+1} + P_n P_p$$

(Bicknell, 1975, p.347).

There is also a relationship between the Pell Sequence and the Pell Equation. If we define the Pell Equation as

$$x^2 + 2y^2 = \pm 1$$

And if

$$x = p_{n+1} - p_n$$

And

$$y = p_n,$$

Then x and y will satisfy the Pell Equation (Ayoub, 2002). In other words, if we take any two consecutive terms of the Pell Sequence, then their difference and the smaller one will satisfy one of the Pell Equations (Ayoub, 2002).

- The proportion $\sqrt{2}:1$ or $\frac{99}{70}$ is used in paper sizes A3, A4 and others. For example, the A4 paper size is 3 times $\frac{99}{70}$ or $297 \times 210\ mm$ or the standard 11.7×8.3 which is more commonly known as 8.5×11 paper (Michell & Brown, 2009, p.133).

4.2: Solving the Pell sequence using 4 techniques

In this section I will use the 4 techniques from Chapter 2 to solve the Pell Sequence: Guess and Check with Induction, Characteristic Polynomial, Generating Functions and Linear Algebra.

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1) The Characteristic Polynomial

The Pell Sequence is defined by $p_0 = 1$ and $p_1 = 2$ with $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$ which generates a sequence of $1,2,5,12,29,70,169,408,\ldots$ (Goodaire & Parmenter, 2006, p.182).

Solution:

$$p_n = 2p_{n-1} + p_{n-2} \to p_n - 2p_{n-1} - p_{n-2} = 0 \to$$
$$x^2 - 2x - 1 = 0 \to a = 1, b = -2, c = -1$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \left(1 \pm \sqrt{2}\right)$$
$$x_1 = \left(1 + \sqrt{2}\right) x_2 = \left(1 - \sqrt{2}\right)$$
$$p_n = c_1\left(x_1^n\right) + c_2\left(x_2^n\right) \Rightarrow p_n = c_1\left(1+\sqrt{2}\right)^n + c_2\left(1-\sqrt{2}\right)^n$$
$$p_0 = 1 \to 1 = c_1\left(1+\sqrt{2}\right)^0 + c_2\left(1-\sqrt{2}\right)^0 \to 1 = c_1 + c_2 \leftarrow equation 1$$
$$p_1 = 2 \to 2 = c_1\left(1+\sqrt{2}\right)^1 + c_2\left(1-\sqrt{2}\right)^1 \to 2 = c_1\left(1+\sqrt{2}\right) + c_2\left(1-\sqrt{2}\right) \leftarrow equation 2$$

Multiply equation 1 by $-\left(1+\sqrt{2}\right)$ and then add to equation 2 resulting in:

$$c_1 = \frac{2+\sqrt{2}}{4}.$$

Then substitute c_1 into equation 1 resulting in

$$c_2 = \frac{2-\sqrt{2}}{4}$$

$$\therefore \quad p_n = \frac{2+\sqrt{2}}{4}\left(1+\sqrt{2}\right)^n + \frac{2-\sqrt{2}}{4}\left(1-\sqrt{2}\right)^n \quad \forall\, n \geq 0.$$

A better formula or shortened version would be

$$p_n \approx \left(\frac{2+\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^n \quad \forall\ n \geq 0$$

Which will give us an answer to the closest integer in the Pell Sequence (Goodaire & Parmenter, 2006, p.182).

---

2) Generating Functions

The Pell Sequence is defined by $p_0 = 1, p_1 = 2, p_2 = 5$ and

$$p_n = 2p_{n-1} + p_{n-2} \rightarrow p_n - 2p_{n-1} - p_{n-2} = 0$$

Solution:

$$f(x) = \sum_{n=0}^{\infty} a_n x_n$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

$$2xf(x) = 2a_0 x + 2a_1 x^2 + \ldots + a_{n-1} x^n$$

$$x^2 f(x) = a_0 x^2 + \ldots + a_{n-2} x^n$$

$$f(x) - 2xf(x) - x^2 f(x) =$$

$$a_0 + \left(a_1 - 2a_0\right)x + \left(a_2 - 2a_1 - a_0\right)x^2$$

$$1 + \left(2 - 2(1)\right)x + \left(5 - 2(2) - 1\right)x^2 = 1$$

$$f(x)\left(1 - 2x - x^2\right) = 1 \rightarrow f(x) = \frac{1}{1 - 2x - x^2}$$

$$a = -1, b = -2, c = 1 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-1)(1)}}{2(-1)} \rightarrow x = -1 \pm \sqrt{2}$$

$$\frac{A}{\left[\left(-1-\sqrt{2}\right) - x\right]} + \frac{B}{\left[\left(-1+\sqrt{2} - x\right)\right]} = \frac{1+0x}{1 - 2x - x^2}$$

$$A\left[\left(-1+\sqrt{2}\right)-x\right]+B\left[\left(-1-\sqrt{2}\right)-x\right]=1+0x$$
$$A\left(-1+\sqrt{2}\right)-Ax+B\left(-1-\sqrt{2}\right)-Bx=1+0x$$
$$equation1\rightarrow -A-B=0$$
$$equation2\rightarrow A\left(-1+\sqrt{2}\right)+B\left(-1-\sqrt{2}\right)=1$$

Multiply equation 1 by $\left(-1+\sqrt{2}\right)$ and add to equation 2.

$$-B\left(-1+\sqrt{2}\right)+B\left(-1-\sqrt{2}\right)=1$$
$$-B\left[\left(-1+\sqrt{2}\right)-\left(-1-\sqrt{2}\right)\right]=1$$
$$-B\left(2\sqrt{2}\right)=1\rightarrow B=-\frac{1}{2\sqrt{2}}=-\frac{\sqrt{2}}{4}$$

Substitute B back into equation 1:

$$A=\frac{\sqrt{2}}{4}$$
$$\frac{\sqrt{2}}{4}\left(\frac{1}{\left(-1-\sqrt{2}\right)-x}\right)-\frac{\sqrt{2}}{4}\left(\frac{1}{\left(-1+\sqrt{2}\right)-x}\right)$$
$$sidenote$$
$$\frac{1}{a-x}=\left(\frac{1/a}{1/a}\right)=\left(\frac{1/a}{1-\left(x/a\right)}\right)=\frac{1}{a}\left(\frac{1}{1-\left(x/a\right)}\right)=$$
$$\frac{1}{a}\left(1+\left(\frac{x}{a}\right)+\left(\frac{x}{a}\right)^2+\left(\frac{x}{a}\right)^3+\ldots\right)=\left(\frac{1}{a}\right)\bullet a^{-n}$$
$$sidenote$$
$$=\frac{\sqrt{2}}{4}\left(\frac{1}{-1-\sqrt{2}}\right)\left(\frac{1}{-1-\sqrt{2}}\right)^n-\frac{\sqrt{2}}{4}\left(\frac{1}{-1+\sqrt{2}}\right)\left(\frac{1}{-1+\sqrt{2}}\right)^n$$
$$=\frac{\sqrt{2}}{4}\left(\frac{-1+\sqrt{2}}{-1}\right)\left(\frac{-1+\sqrt{2}}{-1}\right)^n-\frac{\sqrt{2}}{4}\left(\frac{-1-\sqrt{2}}{-1}\right)\left(\frac{-1-\sqrt{2}}{-1}\right)^n$$
$$=\frac{\sqrt{2}}{4}\left(1-\sqrt{2}\right)^{n+1}-\frac{\sqrt{2}}{4}\left(1+\sqrt{2}\right)^{n+1}$$
$$=\frac{\sqrt{2}}{4}\left[\left(1-\sqrt{2}\right)^{n+1}-\left(1+\sqrt{2}\right)^{n+1}\right],\forall n\geq 0$$

3) Linear Algebra

The Pell Sequence can also be defined by $p_0 = 0$ and $p_1 = 1$ with

$p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$ which generates a sequence of

$$0,1,2,5,12,29,70,169,408,\ldots$$

(Burton, 2002, p.332)

Solution: For the linear algebra approach, we phrase it in matrix terms:

$$\begin{bmatrix} p_{n+1} \\ p_n \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n \\ p_{n-1} \end{bmatrix}$$
$$v_n = A^n \bullet v_{n-1}$$
$$A^n v_0 = A^n \begin{bmatrix} p_1 \\ p_0 \end{bmatrix} = A^n \begin{bmatrix} 2 \\ 1 \end{bmatrix} \leftarrow v_0$$

Next, we will find the characteristic polynomial by the diagonalization of A.

$$(A - \lambda I) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$
$$\det(A - \lambda I) = (2-\lambda)(-\lambda) - (1)(1) \rightarrow \lambda^2 - 2\lambda - 1$$
$$a = 1, b = -2, c = -1 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} \rightarrow \frac{2 \pm \sqrt{8}}{2} \rightarrow \frac{2 \pm 2\sqrt{2}}{2}$$
$$x = 1 \pm \sqrt{2} \rightarrow \lambda_1 = 1 + \sqrt{2}, \lambda_2 = 1 - \sqrt{2}$$

The numbers λ_1 and λ_2 are Eigen values of A and $\lambda_1 = 1 + \sqrt{2}, \lambda_2 = 1 - \sqrt{2}$. To find the Eigen space for $\lambda_1 = 1 + \sqrt{2}$ we do the following:

$$\begin{bmatrix} 2-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$(2 - \lambda_1)x_1 + x_2 = 0$$
$$x_1 - \lambda_1 x_2 = 0$$

Where $x_2 = t$ is free. This leads to

$$x_1 = \lambda_1 x_2 = \lambda_1 t \to x = \begin{bmatrix} \lambda_1 t \\ t \end{bmatrix} = t \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

To find the Eigen space for $\lambda_2 = 1 - \sqrt{2}$ we do the following:

$$\begin{bmatrix} 2-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(2-\lambda_2)x_1 + x_2 = 0$$

$$x_1 - \lambda_2 x_2 = 0$$

Where $x_2 = t$ is free. This leads to

$$x_1 = \lambda_2 x_2 = \lambda_2 t = x = \begin{bmatrix} \lambda_2 t \\ t \end{bmatrix} = t \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

Next, we define

$$P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \quad \text{And} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

And we compute that

$$P^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

Then, the powers of A can be computed using this diagonal form:

$$PD^n = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} = \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix}$$

$$v_n = PD^n = \frac{1}{2\sqrt{2}} \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \begin{bmatrix} 2-\lambda_2 \\ -2+\lambda_1 \end{bmatrix}$$

$$v_n = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2\lambda_1^{n+1} - \lambda_2\left(\lambda_1^{n+1}\right) + \left(-2\lambda_2^{n+1}\right) + \lambda_2^{n+1}\left(\lambda_1\right) \\ 2\lambda_1^n - \lambda_2\left(\lambda_1^n\right) + \left(-2\lambda_2^n\right) + \lambda_2^n\left(\lambda_1\right) \end{bmatrix} \begin{matrix} a_{n+1} \\ a_n \end{matrix}$$

$$p(n) = \frac{1}{2\sqrt{2}} \left[\lambda_1^n\left(2-\lambda_2\right) + \lambda_2^n\left(-2+\lambda_1\right)\right]$$

$$p(n) = \frac{1}{2\sqrt{2}} \left[\lambda_1^n\left(2-\left(1-\sqrt{2}\right)\right) + \lambda_2^n\left(-2+1+\sqrt{2}\right)\right]$$

$$p(n) = \frac{1}{2\sqrt{2}} \left[\left(1+\sqrt{2}\right)^n\left(1+\sqrt{2}\right) + \left(1-\sqrt{2}\right)^n\left(-1+\sqrt{2}\right)\right]$$

$$p(n) = \left[\left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\left(\frac{-1+\sqrt{2}}{2\sqrt{2}}\right)\right]$$

$$p(n) = \left[\left(\frac{\sqrt{2}+2}{4}\right)\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\left(\frac{-\sqrt{2}+2}{4}\right)\right]$$

---

4) Guess and Check with Induction

The desired sequence is defined recursively by $p(n) = 2p_{n-1} + p_{n-2}$ where $p_0 = 0$ and $p_1 = 1$ (Burton, 2002, p.332). The first 9 terms are $0,1,2,5,12,29,70,169,408,\ldots$ A formula that was found using the characteristic polynomial is

$$p(n) = \left(\frac{\sqrt{2}}{4}\right)\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right] \text{ For all } n \geq 0.$$

As is always the case when we wish to verify a formula for a recursively-defined sequence, we can attempt to do so using mathematical induction.

(i) **Base case(s).** When $n = 0$

$$p(n) = \left(\frac{\sqrt{2}}{4}\right)\left[\left(1+\sqrt{2}\right)^{0} - \left(1-\sqrt{2}\right)^{0}\right] = 0$$

And when $n = 1$

$$p(n) = \left(\frac{\sqrt{2}}{4}\right)\left[\left(1+\sqrt{2}\right)^{1} - \left(1-\sqrt{2}\right)^{1}\right] = 1$$

(ii) **Induction Step**: (Burton, 2002, p.332) To show

$$p_n = \frac{\sqrt{2}}{4}\left[\left(1+\sqrt{2}\right)^{n} - \left(1-\sqrt{2}\right)^{n}\right] \text{ Or } \frac{\sqrt{2}}{4}\left(\alpha^{n} - \beta^{n}\right)$$

Where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$

Assume: $p_n = \dfrac{\sqrt{2}}{4}\left(\alpha^{n} - \beta^{n}\right)$ for $0 \le n \le k$

Then

$$2p_{n-1} + p_{n-2} = 2\left[\frac{\sqrt{2}}{4}\left(\alpha^{n-1} - \beta^{n-1}\right)\right] + \left[\frac{\sqrt{2}}{4}\left(\alpha^{n-2} - \beta^{n-2}\right)\right]$$
$$= \frac{\sqrt{2}}{4}\left[2\alpha^{n-1} - 2\beta^{n-1} + \alpha^{n-2} - \beta^{n-2}\right]$$
$$= \frac{\sqrt{2}}{4}\left[2\alpha^{n-1} + \alpha^{n-2} - 2\beta^{n-1} - \beta^{n-2}\right]$$
$$= \frac{\sqrt{2}}{4}\left[\alpha^{n-2}(2\alpha+1) - \beta^{n-2}(2\beta+1)\right]$$

Side note:

$$2\alpha + 1 \rightarrow 2\left(1+\sqrt{2}\right) + 1 = 2 + 2\sqrt{2} + 1 = 3 + 2\sqrt{2}$$
$$\alpha^{2} \rightarrow \left(1+\sqrt{2}\right)^{2} = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$$

Back to the proof:

Since $\alpha^{2} = 2\alpha + 1$ and $\beta^{2} = 2\beta + 1$ then,

$$= \frac{\sqrt{2}}{4}\left[\alpha^{n-2}\alpha^2 - \beta^{n-2}\beta^2\right]$$
$$= \frac{\sqrt{2}}{4}\left[\alpha^n - \beta^n\right]$$
$$= p_n$$

4.3: An Alternate Explicit Formula for the Pell Sequence

The Mathematical Association of America (MAA) publishes a popular small journal called the *American Mathematical Monthly*. In it, (AMM, April 2000, p.370-371) they describe a fascinating alternate explicit formula for the Pell Sequence. Below is the formula as well as three proofs verifying the formula.

10663 [1998, 464]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* For all $n \geq 0$, the Pell numbers are given by the expression

$$p_n = \sum \frac{(i+j+k)!}{i!\,j!\,k!}$$

Where the summation extends over all nonnegative integers i, j, k satisfying $i + j + 2k = n$.

Solution I *by Harris Kwong, State University of New York, Fredonia, NY.*

Let p_n count the ways to fill a n-foot flagpole. The colors we will use to represent this are red, white, and blue flags. The white flags are two feet tall, blue are 1 feet tall, and red are 1 feet tall. We will equate i, j, k with each flag, $red = i$,

$blue = j$, and $white = k$. This yields $p_0 = 1, p_1 = 2$ and for $n \geq 2$ the recurrence follows by looking at the options for the last flag.

To fill the pole with i red flags, j blue flags, and k white flags, we will use the condition that $i + j + 2k = n$. The number of ways to arrange these flags is $\frac{(i+j+k)!}{i!j!k!}$. If we sum over the choices of i, j, k with $i + j + 2k = n$ results in p_n (Kwong, April 2000,AMM, p.370).

If all of the flags are 1 foot flags, meaning all blue, all red, or any combination of the two, then there are 3^n possibilities or $3^6 = 729$ possibilities. Another situation arises if we consider for all cases what flag is on top of the flagpole. **Case 1**: If a blue flag is on top then anything underneath it will be $p_n - 1$. **Case 2**: If a red flag is on top then once again anything underneath it is $p_n - 1$. **Case 3**: If it is a white flag then anything underneath it will be $p_n - 2$. This yields the desired recurrence relation, $p_n = 2p_{n-1} + p_{n-2}$ where $2p_{n-1} = r, b$ and $p_{n-2} = w$. Below are some examples on how this would work on a case-by-case basis.

1) When $n = 0$ then $p_0 = 1$ and the only triple i, j, k that satisfies

$i + j + 2k = 0$ Is $(0,0,0)$.

There is only one way to have a zero foot flagpole if all flags are zero feet tall.

2) When $n = 1$ then $p_1 = 2$ and two triples i, j, k satisfy

$i + j + 2k = 1 \rightarrow (1,0,0)$ And $(0,1,0)$

There are only two ways to fill a one-foot flagpole either with one blue flag or one red flag.

3) When $n = 2$ then $p_2 = 5$ and four triples i, j, k satisfy $i + j + 2k = 2$, they are $(2,0,0),(0,2,0),(0,0,1),(1,1,0)$ with $(1,1,0)$ having two options either blue or red on top. There are five ways to fill a two-foot flagpole. The options are: a) two red, zero blue, zero white. b) One red, one blue, zero white. c) Zero red, two blue, zero white. d) Zero red, zero blue, one white. e) One blue, one red, zero white. All other n-foot flagpoles will have a similar pattern.

Solution II *by Cecil C. Rousseau, University of Memphis, Memphis, TN.*

Recall, the Binomial Theorem states:

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} = \sum_{\substack{k,j \\ k+j=n}} \frac{n!}{k!\,j!} A^k B^j .$$

We can do a trinomial expansion on the Binomial Theorem:

$$(A+B+C)^n = \sum_{\substack{i,j,k \\ i+j+k=n}} \frac{n!}{i!\,j!\,k!} A^i B^j C^k .$$

A geometric sequence is represented by

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots + r^n .$$

Let

$$G(z) = \sum_{n \geq o} p_n z^n .$$

Next, multiply the recurrence by z^n and sum over $n \geq 2$ produces,

$$G(z) - 1 - 2z = 2z\big(G(z) - 1\big) + z^2 G(z)$$

Then, doing a trinomial expansion results in:

$$G(z)=\frac{1}{1-2z-z^2}=\frac{1}{1-\left(2z+z^2\right)}=1+\left(2z+z^2\right)+\left(2z+z^2\right)^2+\ldots+\left(2z+z^2\right)^m$$

$$=\sum_{m\geq 0}\left(z+z+z^2\right)^m=\sum_{i,j,k\geq 0}\frac{(i+j+k)!}{i!\,j!\,k!}z^i z^j z^{2k}$$

Collecting contributions to the coefficients of z^n completes the proof (Rousseau, April 2000, AMM, p.370-371).

Solution III *by Paul K. Stockmeyer, College of William and Mary, Williamsburg, VA.*

If we replace j with $n-i-2k$ yields the summand,

$$F(n,i,k)=\frac{(n-k)!}{i!\,(n-i-2k)!\,k!}$$

For $i\geq 0, k\geq 0,$ and $n-i-2k\geq 0,$ with $F(n,i,k)=0$ otherwise. Then we want to show that $p_n=f(n)$ for $n\geq 0$, where $f(n)=\sum_{i,k}F(n,i,k)$, with the sum taken over all integer values of i and k. Below is the mathematical computation.

$$F(n,i,k)=F(n-1,i,k)+F(n-1,i-1,k)+F(n-2,i,k-1)$$

For $n\geq 2$ and all i and k. Summing this over all i and k produces,

$$f(n)=f(n-1)+f(n-1)+f(n-2),$$

This can be verified with $f(0)=p_0$ and $f(1)=p_1$. (Stockmeyer, April 2000, American Mathematical Monthly, p.370-371)

I have calculated the first 5 terms of the sequence by using the alternate explicit formula below.

$$p_n=\sum_{\substack{i,j,k\geq 0\\ i+j+2k=n}}\frac{(i+j+k)!}{i!\,j!\,k!}$$

$$p_0 = 1$$
$$i + j + 2k = n$$
$$i + j + 2k = 0$$
$$(i, j, k)$$
$$(0,0,0)$$
$$\frac{(0+0+0)!}{0!0!0!} = \frac{1}{1} = 1$$

$$p_1 = 2$$
$$i + j + 2k = n$$
$$i + j + 2k = 1$$
$$(i, j, k)$$
$$(0,1,0) + (1,0,0)$$
$$\frac{(0+1+0)!}{0!1!0!} + \frac{(1+0+0)!}{1!0!0!}$$
$$\frac{1}{1} + \frac{1}{1} = 1 + 1 = 2$$

$$p_2 = 5$$
$$i + j + 2k = n$$
$$i + j + 2k = 2$$
$$(i, j, k)$$
$$(2,0,0) + (1,1,0) + (0,2,0) + (0,0,1)$$
$$\frac{(2+0+0)!}{2!0!0!} + \frac{(1+1+0)!}{1!1!0!} + \frac{(0+2+0)!}{0!2!0!} + \frac{(0+0+1)!}{0!0!1!}$$
$$\frac{2}{2} + \frac{2}{1} + \frac{2}{2} + \frac{1}{1} = 1 + 2 + 1 + 1 = 5$$

$p_3 = 12$
$i + j + 2k = n$
$i + j + 2k = 3$
(i, j, k)
$(0,1,1) + (1,0,1) + (2,1,0) + (1,2,0) + (3,0,0) + (0,3,0)$

$$\frac{(0+1+1)!}{0!1!1!} + \frac{(1+0+1)!}{1!0!1!} + \frac{(2+1+0)!}{2!1!0!} + \frac{(1+2+0)!}{1!2!0!} + \frac{(3+0+0)!}{3!0!0!} + \frac{(0+3+0)!}{0!3!0!}$$

$2 + 2 + 3 + 3 + 1 + 1 = 12$

$p_4 = 29$
$i + j + 2k = n$
$i + j + 2k = 4$
(i, j, k)
$(0,2,1) + (2,0,1) + (1,1,1) + (0,0,2) + (1,3,0) + (3,1,0) + (4,0,0) + (0,4,0) + (2,2,0)$

$$\frac{(0+2+1)!}{0!2!1!} + \frac{(2+0+1)!}{2!0!1!} + \frac{(1+1+1)!}{1!1!1!} + \frac{(0+0+2)!}{0!0!2!} + \frac{(1+3+0)!}{1!3!0!} + \frac{(3+1+0)!}{3!1!0!} \frac{(4+0+0)!}{4!0!0!} + \frac{(0+4+0)!}{0!4!0!}$$
$$+ \frac{(2+2+0)!}{2!2!0!}$$

$3 + 3 + 6 + 1 + 4 + 4 + 1 + 1 + 6 = 29$

4.4: Pell and Lucas Numbers: Binet formulas and Pell Identities

In this section we consider two related versions of the Pell Numbers. The first version, which gives the traditional Pell Numbers, or Pell Sequence, are defined by

$$p_0 = 0,\ p_1 = 1,$$

$$p_n = 2p_{n-1} + p_{n-2},\ n \geq 2$$

And the second version having the conditions of

$$q_0 = 1,\ q_1 = 1,$$

$$q_n = 2q_{n-1} + q_{n-2},\ n \geq 2.$$

(Burton, 2002, p.332)

This gives us two sequences

$$0,1,2,5,12,29,70,169,408,\ldots$$
$$1,1,3,7,41,99,239,577,\ldots$$

If $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ we can show that the Pell numbers can be expressed as

$$p_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \qquad q_n = \frac{\alpha^n + \beta^n}{2}$$

For $n \geq 0$ (Burton, 2002, p.332).

As we have seen, given the recurrence relation

$$p_n = 2p_{n-1} + p_{n-2},$$

We can bring the sequence all to the left side and set it equal to zero as such,

$$p_n - 2p_{n-1} - p_{n-2} = 0.$$

From this we get the quadratic equation $x^2 - 2x - 1 = 0$. Using the quadratic formula we get the roots of

$$\alpha = 1+\sqrt{2} \quad \text{And} \quad \beta = 1-\sqrt{2}.$$

Substituting α and β in for x we get $\alpha^2 - 2\alpha - 1 = 0$ and $\beta^2 - 2\beta - 1 = 0$. From here we will set the second and third term equal to the first term of each equation giving us $\alpha^2 = 2\alpha + 1$ and $\beta^2 = 2\beta + 1$.

Next, we will multiply the corresponding equations by α^n and β^n resulting in $\alpha^{n+2} = 2\alpha^{n+1} + \alpha^n$ and $\beta^{n+2} = 2\beta^{n+1} + \beta^n$. Subtracting the second equation from the first and dividing by $\alpha - \beta$ gives us

$$\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

If we set

$$H_n = \frac{(\alpha^n - \beta^n)}{(\alpha - \beta)}$$

We get $H_{n+2} = 2H_{n+1} + H_n$. Notice that $H_1 = 1$ and

$$H_2 = \frac{(\alpha^2 - \beta^2)}{(\alpha - \beta)} = \alpha + \beta = 2.$$

Therefore the sequence H_n is the sequence of the Pell numbers p_n,

$$\Rightarrow p_n = \frac{(\alpha^n - \beta^n)}{(\alpha - \beta)} = \frac{(\alpha^n - \beta^n)}{2\sqrt{2}}.$$

For the second sequence, everything will be the same as above but will yield

$$\Rightarrow q_n = \frac{(\alpha^n + \beta^n)}{(\alpha + \beta)} = \frac{(\alpha^n + \beta^n)}{2}.$$

(Burton & Paulsen, 2002, p.140).

Having now established these expressions, I will derive five relations in regards to the Pell numbers from the two sequences given above.

a) $p_{2n} = 2p_n q_n$

b) $p_n + p_{n-1} = q_n$

c) $2q_n^2 - q_{2n} = (-1)^n$

d) $p_n + p_{n+1} + p_{n+3} = 3p_{n+2}$

e) $q_n^2 - 2p_n^2 = (-1)^n$; Hence q_n / p_n are the convergents of $\sqrt{2}$

(Burton, 2002, p.332)

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a) $p_{2n} = 2p_n q_n$.

Solution:

The closed form for the Pell numbers is as follows:

$$p_n = \left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^n + \left(-\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^n$$

This gives us the sequence of Pell numbers

$$0,1,2,5,12,29,70,169,408,\ldots$$

And

$$q_n = \left(\frac{1}{2}\right)\left[\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right]$$

Gives us a different sequence of Pell numbers of

$$1,1,3,7,17,41,99,239,577,\ldots$$

Using the following relation of $p_{2n} = 2p_n q_n$ gives us:

$$\left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{2n} + \left(-\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{2n} - 2\left[\left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^n + \left(-\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^n\right]$$
$$\left[\left(\frac{1}{2}\right)\left(1+\sqrt{2}\right)^n\left(1-\sqrt{2}\right)^n\right]$$

Factoring out a 2 gives us the next result:

$$2\left[\left(\frac{\sqrt{2}}{8}\right)\left(1+\sqrt{2}\right)^{2n} + \left(\frac{\sqrt{2}}{8}\right)\left(1+\sqrt{2}\right)^n\left(1-\sqrt{2}\right)^n + \left(-\frac{\sqrt{2}}{8}\right)\left(1-\sqrt{2}\right)^{2n} + \left(-\frac{\sqrt{2}}{8}\right)\left(1+\sqrt{2}\right)^n\left(1-\sqrt{2}\right)^n\right]$$

Next we simplify by collecting like terms:

$$2\left[\left(\frac{\sqrt{2}}{8}\right)\left(1+\sqrt{2}\right)^{2n} + \left(-\frac{\sqrt{2}}{8}\right)\left(1-\sqrt{2}\right)^{2n}\right]$$

We then distribute the 2 back into the brackets and reduce giving the desired result:

$$\left[\left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{2n} + \left(-\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{2n}\right]$$

b) $p_n + p_{n-1} = q_n$

Solution: Rearranging the relationship gives us $q_n = p_n + p_{n-1}$

$$\left(\frac{1}{2}\right)\left[\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right] =$$
$$\left[\left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^n + \left(-\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^n\right] + \left[\left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{n-1} + \left(-\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{n-1}\right]$$

Next we arrange the terms to have corresponding parts:

$$\left[\left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^n + \left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{n-1} + \left(-\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^n + \left(-\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{n-1}\right]$$

Then we factor to obtain the following:

$$\left(\frac{\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{n-1}\left[\left(1+\sqrt{2}+1\right)\right] + \left(-\frac{\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{n-1}\left[\left(1-\sqrt{2}\right)+1\right]$$

We then distribute back through the brackets to obtain the following:

$$\left(\frac{\sqrt{2}}{4}\right)\left(2+\sqrt{2}\right)\left(1+\sqrt{2}\right)^{n-1} + \left(-\frac{\sqrt{2}}{4}\right)\left(2-\sqrt{2}\right)\left(1-\sqrt{2}\right)^{n-1}$$

Then distribute, collect like terms and simplify:

$$\left(\frac{1+\sqrt{2}}{2}\right)\left(1+\sqrt{2}\right)^{n-1} + \left(\frac{1-\sqrt{2}}{2}\right)\left(1-\sqrt{2}\right)^{n-1}$$

Then we factor out $\frac{1}{2}$ and simplify getting the desired result:

$$\left(\frac{1}{2}\right)\left[\left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right)^{n-1} + \left(1-\sqrt{2}\right)\left(1-\sqrt{2}\right)^{n-1}\right]$$
$$\left(\frac{1}{2}\right)\left[\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right]$$

c) $2q_n^2 - q_{2n} = (-1)^n$

Solution: Rearranging the relationship gives us $(-1)^n = 2q_n^2 - q_{2n}$

$$(-1)^n = 2\left[\left(\frac{1}{2}\right)\left[\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right]\right]\left[\left(\frac{1}{2}\right)\left[\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right]\right] - \left[\left(\frac{1}{2}\right)\left[\left(1+\sqrt{2}\right)^{2n} + \left(1-\sqrt{2}\right)^{2n}\right]\right]$$

Next we will distribute through and collect like terms:

$$\left[\left(\frac{1}{2}\right)\left(1+\sqrt{2}\right)^{2n} + \left(1+\sqrt{2}\right)^n\left(1-\sqrt{2}\right)^n + \left(\frac{1}{2}\right)\left(1-\sqrt{2}\right)^{2n}\right] - \left[\left(\frac{1}{2}\right)\left[\left(1+\sqrt{2}\right)^{2n} + \left(1-\sqrt{2}\right)^{2n}\right]\right]$$

Then we distribute through $\frac{1}{2}$ and eliminate some brackets:

$$\left[\left(\frac{1}{2}\right)\left(1+\sqrt{2}\right)^{2n} + \left(1+\sqrt{2}\right)^n\left(1-\sqrt{2}\right)^n + \left(\frac{1}{2}\right)\left(1-\sqrt{2}\right)^{2n} - \left(\frac{1}{2}\right)\left(1+\sqrt{2}\right)^{2n} - \left(\frac{1}{2}\right)\left(1-\sqrt{2}\right)^{2n}\right]$$

Next we collect like terms and reduce:

$$\left(\left(1+\sqrt{2}\right)^n\left(1-\sqrt{2}\right)^n\right) \rightarrow \left(\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)\right)^n$$

Finally we distribute the two binomials and obtain the desired result:

$$(-1)^n$$

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d) $p_n + p_{n-1} + p_{n-3} = 3p_{n-2}$

Solution: We begin with the recurrence:

$$\begin{aligned}
&p_n = 2p_{n-1} + p_{n-2} \rightarrow p_n - 2p_{n-1} - p_{n-2} = 0 \\
&p_{n+3} = 2p_{n+2} + p_{n+1} \leftarrow equation 1 \\
&p_{n+2} = 2p_{n+1} + p_n \rightarrow p_{n+1} + p_n = p_{n+2} \leftarrow equation 2
\end{aligned}$$

Adding equation 1 to equation 2 results in:

$$p_n + 2p_{n+1} + p_{n+3} = 3p_{n+2} + p_{n+1}$$

Adding $-p_{n+1}$ to both sides results in:

$$p_n + p_{n+1} + p_{n+3} = 3p_{n+2}$$

---

e) $q_n^2 - 2p_n^2 = (-1)^n$

Solution: Note from earlier that:

$$p_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \text{ And } q_n = \frac{\alpha^n + \beta^n}{2}$$

Where

$$\alpha = 1 + \sqrt{2}, \beta = 1 - \sqrt{2}$$

(Burton, 2002, p.332)

With the identity $q_n^2 - 2p_n^2 = (-1)^n$ given to us above, q_n^2 equals the following:

$$q_n^2 = \frac{(\alpha^n + \beta^n)^2}{2^2} = \frac{\alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n}}{4}$$

Next p_n^2 equals the following:

$$p_n^2 = \frac{(\alpha^n - \beta^n)^2}{(2\sqrt{2})^2} = \frac{\alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n}}{8}$$

Then we take the difference of each above multiply p_n^2 by two:

$$q_n^2 - 2p_n^2 = \frac{\alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n}}{4} - 2\left(\frac{\alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n}}{8}\right) = \frac{4\alpha^n\beta^n}{4}$$

Then,

$$\frac{4\alpha^n\beta^n 4}{4} = \alpha^n\beta^n = (\alpha\beta)^n$$

Which will then give us the desired result:

$$\left[(1+\sqrt{2})(1-\sqrt{2})\right] = (1+\sqrt{2}-\sqrt{2}-2)^n = (-1)^n$$

Chapter 5: Curriculum for Instructors and Students

5.1: About the Curriculum

The curriculum component of my M.S.T. was done in my period 6, first semester school year 2010-2011 Algebra II class. The class has about 25 students and it has all high school students. It had one 9th grader, 2 seniors, a few 11th graders with predominately 10th graders. I have taught at my school Arts and Communication Magnet Academy (A.C.M.A.) for 8 years. A.C.M.A. is a grade 6 through 12 public arts magnet school in the Beaverton, Oregon school district. Many of the students in this Algebra class I have known since 6th grade, so we are very familiar with each other. These lessons were done during the month of January 2011 over about 4 weeks. We met about 10 class periods during this time. Five lessons were used with the students: Introduction to Recurrence Relations, Characteristic Polynomial, Guess and Check with Induction parts 1 and 2, Pell Sequence, Tower of Hanoi. The Back Substitution lesson and Flagpoles lesson were not done with the class due to time constraints. They were tested and simulated with myself. In the appendix below you will see each lesson categorized by: Lesson Plan, Student Handout, Instructor Handout with solutions, and reflection for each lesson.

5.2: Introduction to Recurrence Relations

5.2.1: Lesson Plan

Lesson 1: Introduction to Recurrence Relations

Grade level: High School (9-12)	Subject: Advanced Algebra II	Unit Title: Techniques for creating explicit formulas from recurrence relations	Unit topic: Discrete Math-Recurrence Relations
Lesson# One	Lesson Title: Introduction to recurrence relations	Materials: Handout Advanced Algebra II textbook Graphing calculator	Overview and purpose: Prerequisite material and an introduction to recurrence relations

Objectives: The purpose of this lesson is to introduce students to recurrence relations. This lesson is a prerequisite so students can be prepared for more complex lessons to come.
Information: Through modeling, an investigative handout, as well as their textbook, students will become familiar with various definitions in recursion, arithmetic and geometric sequences, non-geometric and non-arithmetic sequences, writing recursive formulas, writing terms in a sequence from a recursive formula, shifted geometric sequences and applications. In order for students to have an easier transition to the harder material to follow, it is important that the class cover a chapter or unit on recursion. Usually a high school Advanced Algebra II, Precalculus, or Discrete Math textbook will suffice. Topics that should be covered but are not limited to are arithmetic and geometric sequences: their graphs and applications, shifted geometric sequences (concept of a limit), non-arithmetic and non-geometric sequences, writing recursive formulas, and writing terms in a sequence from a recursive formula.

Verification: Students will check each other for understanding during the class period by having their peers present their findings in front of the classroom. They will teach each other. Teacher will also check for understanding by prompting and promoting discussion and with open-ended questioning. Either at the end of the lesson or the following class period the teacher will collect student work and grade and give appropriate feedback. The following class period the class should review the previous days lesson to ensure comprehension and understanding before moving on to the next lesson.
Activity: With the handout provided and a standard high school textbook with a chapter on recursion, students will go through the handout in a tutorial investigative like approach to complete and comprehend the material.
Summary: At the end of this lesson and unit, students will have a good foundation of recursion. They will be able to understand more complex topics within recursion. More specifically, they will need to convert recurrence relations to an explicit formula by using four techniques: guess and check with induction, characteristic polynomial, generating functions, and linear algebra.

5.2.2: Student Handout

Name_______________________
Period_________

Lesson 1: Introduction to Recurrence Relations

Give definitions for the following words below as well as an example.

1) Recursion

2) Sequence

3) Term

4) General term (Murdock, Kamischkie, & Kamischkie, 2004, p.29)

5) Recurrence relation (recursive formula)

6) Initial condition(s)

7) What is an arithmetic sequence? Give the symbolic form and what each part means.

8) What is a geometric sequence? Give the symbolic form and what each part means.

9) What is the symbolic form of a shifted geometric sequence? What is the long-run value? (Murdock et. al., 2004)

10) Write a recursive formula to generate each sequence. Use u_1 for the first term. Then find the next 3 terms in the sequence.

a) 3, 7, 11, 15, …

b) 15, 5, -5, -15, …

c) .3, .03, .003, .0003, …

d) 100, 150, 225, 337.5, …

11) List the first five terms of the sequence.

a) $u_1 = -4$

$u_n = u_{n-1} - 1.5$ Where $n \geq 2$

b) $u_1 = 1$

$u_n = 3u_{n-1} - 2$ Where $n \geq 2$

c) $u_0 = 256$

$u_n = 0.75\, u_{n-1}$, $n \geq 1$

12) Application: A nursery owns 7000 Japanese maple trees. Each year the nursery plans to sell 12% of it’s trees and plant 600 new ones (Murdock et al., 2004, p.49).

a) Write a recursive definition that represents the nursery’s tree stock.

b) Find the number of pine trees owned by the nursery after 10 years.

c) At what point will the amount of trees planted equal the amount of trees sold by the nursery?

13) Here are some recurrence relations that are neither arithmetic nor geometric. List the first 6 terms of each sequence. Instead of u_n use a_n.

a) $a_1 = \frac{3}{2}$

$a_n = 5a_{n-1} - 1$ For $n \geq 2$

b) $a_0 = -3$, $a_1 = -2$

$a_n = 5a_{n-1} - 6a_{n-2}$ For $n \geq 2$

c) $a_1 = 10$, $a_2 = 29$

$a_{n+1} = 7a_n - 10a_{n-1}$ For $n \geq 2$

14) Write a recurrence relation for the following sequences. Use a_1 for the first term in the sequence.

a) 1,1,2,3,5,8,13, …

b) 1,4,9,16, …

c) 1,2,6,24,...

d) 4,1,3, -2,5, -7,12, -19,31, …

5.2.3: Instructor Solutions

Name______________________
Period_________

Lesson 1: Introduction to Recurrence Relations

Give definitions for the following words below as well as an example.

1) Recursion: A process in which the current step relies on the previous step(s).

2) Sequence: An ordered set of numbers.

3) Term: A number in a sequence.

4) General term: a generic term in a sequence usually denoted by u_n or some other variable (Murdock, 2004, p.29).

5) Recurrence relation (recursive formula): An equation that defines one member of the sequence in terms of a previous one.

6) Initial condition(s): Starting term(s) in a recurrence relation.

7) What is an arithmetic sequence? Give the symbolic form and what each part means.

   A sequence of numbers where each term is determined by adding the same fixed number to the previous one.

   $u_n = u_{n-1} + d$, u_n Is the current term, u_{n-1} is the previous term and d is the common difference (Murdock, 2004, p.31).

8) What is a geometric sequence? Give the symbolic form and what each part means.
   A sequence in which a constant r can be multiplied by each term to get the next term

$u_n = ru_{n-1}$, u_n Is the current term, u_{n-1} is the previous term and r is the common ratio (Murdock, 2004, p.33).

9) What is the symbolic form of a shifted geometric sequence? What is the long-run value?

$u_n = ru_{n-1} + d$

The long-run value is the limit of the sequence (Murdock, 2004, p.33, 47).

10) Write a recursive formula to generate each sequence. Use u_1 for the first term. Then find the next 3 terms in the sequence.

a) 3, 7, 11, 15, ……

$u_n = u_{n-1} + 4$ Where $u_1 = 3$ and $n \geq 2$

The next 3 terms are 19,23,27

b) 15, 5, -5, -15,..

$u_n = u_{n-1} - 10$ Where $u_1 = 15$ and $n \geq 2$

The next 3 terms are -25, -35, and -45

c) .3, .03, .003, .0003,..

$u_n = (.1)u_{n-1}$ Where $u_1 = .3$ and $n \geq 2$

The next 3 terms in the sequence are .00003, .000003, and .0000003

d) 100, 150, 225, 337.5, ……

$u_n = (\frac{3}{2})u_{n-1}$ Where $u_1 = 100$ and $n \geq 2$

The next 3 terms in the sequence are 537.25,759.375,1139.0625

11) List the first five terms of each sequence

a) $u_1 = -4$

$u_n = u_{n-1} - 1.5$ Where $n \geq 2$

$-5.5, -7, -8.5, -10, -11.5$

b) $u_1 = 1$

$u_n = 3u_{n-1} - 2$ Where $n \geq 2$

$1,1,1,1,1$

c) $u_0 = 256$

$u_n = 0.75\, u_{n-1}$

$192, 144, 108, 81, 60, 75$

12) Application: A nursery owns 7000 Japanese maple trees. Each year the nursery plans to sell 12% of it's trees and plant 600 new ones (Murdock et al., 2004, p.49).

a) Write a recursive definition that represents the nursery's tree stock.

$a_n = (.88)a_{n-1} + 600$ Where $a_1 = 7{,}000$ and $n \geq 2$

b) Find the number of pine trees owned by the nursery after 10 years.

5,557 trees

c) At what point will the amount of trees planted equal the amount of trees sold by the nursery? At 5000 trees the amount sold will be equal to amount planted.

13) Here are some recurrence relations that are neither arithmetic nor geometric. List the first 6 terms of each sequence. Instead of u_n use a_n.

a) $a_1 = \frac{3}{2}$

$a_n = 5a_{n-1} - 1$ For $n \geq 2$

$a_2 = \frac{13}{2}, a_3 = \frac{63}{2}, a_4 = \frac{313}{2}, a_5 = \frac{1563}{2}, a_6 = \frac{7813}{2}$

b) $a_0 = -3$, $a_1 = -2$

$a_n = 5a_{n-1} - 6a_{n-2}$ For $n \geq 2$

$a_2 = 8, a_3 = 52, a_4 = 212, a_5 = 748, a_6 = 2,468$

c) $a_1 = 10$, $a_2 = 29$

$a_{n+1} = 7a_n - 10a_{n-1}$ For $n \geq 2$

$a_3 = 103, a_4 = 431, a_5 = 1,987, a_6 = 9,599$

14) Write a recurrence relation for the following sequences. Use a_1 or a_0 for the first term in the sequence

a)1,1,2,3,5,8,13,..

$a_n = a_{n-1} + a_{n-2}$ Where $n \geq 2$ and $a_0 = 1, a_1 = 2$

b) 1,4,9,16,..

$a_n = n^2$ Where $n \geq 1$ and $a_1 = 1$

c) 1,2,6,24, ……

$a_n = n!$ Where $n \geq 1$ and $a_1 = 1$

d) 4,1,3, -2,5, -7,12, -19,31, ……

Where $n \geq 2$ and $a_0 = 4, a_1 = 1$

5.2.4: Lesson Reflection

Lesson 1: Introduction to Recurrence Relations

Lesson 1 was done shortly after completing Chapter 1 in our Advanced Algebra II textbook. In chapter 1 students had been introduced and became familiar with arithmetic sequences, geometric sequences, shifted geometric sequences, the basic concept of a limit, and recursion notation (Murdock et al., 2004). This lesson was a good starting point for my students. It was designed somewhat like a tutorial where in conjunction with the handout students would go through chapter 1 and answer the questions on the handout. In conjunction with chapter 1 in the textbook, lesson 1 was the last time at review or for prerequisites before all the material to come after would be all new.

This lesson went well overall with most of the students. In the definitions with examples section some of the things I noticed with the students are that some only gave a definition but did not give an example and there were some students who did not define what a long run value is. In the second part where they had to write a recursive definition from geometric and arithmetic sequences when it comes to a ratio some students would write a decimal ratio and some would write a fraction ratio. I constantly struggle with students about the difference between fractions and decimals. I always promote fractions and teach and show the kids the difference between an exact answer and an approximation. A fraction answer will be exact whereas a decimal will be an approximation if it is an irrational number. Students will convert fractions many times to decimals because of their overreliance on calculators as well as not feeling confident in the operations of fractions. A good example of this might be the difference between writing a geometric sequence like this $a_n = \frac{1}{3}a_{n-1}$ or $a_n = (.333\overline{3})a_{n-1}$ which the latter of the two will not produce an exact answer. Some other interesting notes include 14a) only a few students initially picked up on how to write the recurrence relation of the Fibonacci sequence. In 14c) all of the students did not pick up on the fact that this can be represented as a factorial. I also had to go over factorials because most if not all had either never done factorials or did not remember doing them any time in the past in math. This lesson was done during one period and then whatever they did not finish was homework and we went over any questions at the beginning of the next period.

5.3: Characteristic Polynomial

5.3.1: Lesson Plan

Lesson 2: Characteristic Polynomial

Grade level: High School (9-12)	Subject: Advanced Algebra II	Unit Title: Techniques for creating explicit formulas from recurrence relations	Unit topic: Discrete Math-Recurrence Relations
Lesson# Two	Lesson Title: Characteristic Polynomial	Materials: Handout Graphing calculator	Overview and purpose: Create an explicit formula using the Characteristic Polynomial

Objectives: The purpose of this lesson is for students to use the technique of the characteristic polynomial to create explicit formulas from recurrence relations.
Information: Students will be given a handout with an example on how to do the characteristic polynomial. The teacher should also model the example on the handout or another example. Students should have a good foundation in quadratic functions in order to be successful with the characteristic polynomial. Students should be familiar with factoring, roots, the quadratic formula, exponents, substitution and elimination. As well as being familiar with the characteristic polynomial students should also understand conceptually that the drawback to recurrence relations is that each term in the sequence is dependent upon the previous term. By creating an explicit formula the students are able to find any nth term in a non-arithmetic or non-geometric sequence.
Verification: Students will check each other for understanding during the class period by having their peers present their findings in front of the classroom. They will teach each other. Teacher will also check for understanding by prompting and promoting discussion and with open-ended questioning. Either at the end of the lesson or the

following class period the teacher will collect student work and grade and give appropriate feedback. The following class period the class should review the previous days lesson to ensure comprehension and understanding before moving on to the next lesson.
Activity: With the handout provided students will go through the handout in a tutorial investigative like approach to complete and comprehend the material. Students should check their explicit formula by checking the initial conditions as well as the next few terms.
Summary: At the end of this unit and lesson students will have a good understanding of the characteristic polynomial. It is also important that students realize conceptually the difference between a recurrence relation and an explicit formula

5.3.2: Student Handout

Name______________________________
Period___________

<u>Lesson 2: Characteristic Polynomial</u>: A technique for converting recurrence relations to an explicit (closed-form) formula.

The characteristic polynomial is a technique used for solving recursively-defined sequences. Recurrence relations are useful when trying to find patterns in number sequences. Remember though, a drawback when using recurrence relations has been given in the example of, "What if I want to find the 100th term in the sequence?" In order to do that you need to know the 99th term, which means you have to do recursion 99 times.

The characteristic polynomial is the first technique you will learn to find what is called a "closed form-formula", or also known as "explicit formula", for a recurrence relation. A "solution" of the recurrence relations is another way of stating the formula. Creating this will allow you to easily find the 100th term of the sequence as well as any other term in the sequence.

<u>Example:</u> Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = -3$, $a_1 = -2$ an $n \geq 2$ (Goodaire & Parmenter, 2006, p.181).

You can think of the recurrence relation in terms of a quadratic expression of the form $ax^2 + bx + c$ where a, b, and c are constants (numbers). In other words a_n is the ax^2 term, $5a_{n-1}$ is the bx term and $6a_{n-1}$ is the c term.

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$-5a_{n-1} + 6a_{n-2}$ $\quad$ $-5a_{n-1} + 6a_{n-2}$ $\quad$ (Subtract from both sides.)

$a_n - 5a_{n-1} + 6a_{n-2} = 0$ (Which converts to) $x^2 - 5x + 6$

$(x-2)(x-3) = 0$ (Which gives us roots of) $x_1 = 2$ and $x_2 = 3$

(We can next substitute our roots into the equation below)

$a_n = c_1(x_1^n) + c_2(x_2^n)$
(Goodaire & Parmenter, 2006, p.171).

$a_n = c_1(2^n) + c_2(3^n)$

With the first initial condition of $a_0 = -3$ we can substitute 0 for n.

$a_n = c_1(2^n) + c_2(3^n)$ (Becomes) $a_0 = c_1(2^0) + c_2(3^0)$ (Which then becomes) $-3 = c_1 + c_2$
(Remember, anything to the zero power is one.)

The second initial condition is $a_1 = -2$ and we can substitute 1 for n.

$a_n = c_1(2^n) + c_2(3^n)$ (Becomes) $a_1 = c_1(2^1) + c_2(3^1)$ (Which then becomes) $-2 = 2c_1 + 3c_2$

(We now have two equations with two variables.)

$-3 = c_1 + c_2$
$-2 = 2c_1 + 3c_2$

(Solve using elimination or substitution.)

$c_1 = -7$ And $c_2 = 4$

(Substituting) $c_1 = -7$ and $c_2 = 4$ (into) $a_n = c_1(2^n) + c_2(3^n)$ (yields)

$a_n = -7(2^n) + 4(3^n)$ (This is the closed-form/explicit formula, or our solution to the recurrence relation.)

If you test the formula you will see $a_0 = -3$, $a_1 = -2$ and the 100th term is 2061510083000000000000000000000000000000000000000000!

<u>Exercises:</u> For each recurrence relation find the explicit formula. After you create your formula make sure and test the initial conditions to see if it works.

1) $a_n = 4a_{n-1}$ Given $a_0 = 1$, $a_1 = 4$ where $n \geq 2$

2) $a_n = -2a_{n-1} + 15a_{n-2}$ Given $a_0 = 1$, $a_1 = 1$ where $n \geq 2$ (Goodaire & Parmenter, 2006, p.174).

3) $a_n = -6a_{n-1} + 7a_{n-2}$ Given $a_0 = 32$, $a_1 = -17$ (Goodaire & Parmenter, 2006, p.174).

4) $a_n = -8a_{n-1} - a_{n-2}$ Given $a_0 = 0$ and $a_1 = 1$ (Goodaire & Parmenter, 2006, p.174).

5) $a_{n+1} = 7a_n - 10a_{n-1}$ Given $a_1 = 10$ and $a_2 = 29$ (Goodaire & Parmenter, 2006, p.174).

5.3.3: Instructor Solutions

Name_______________________________

Period__________

Lesson 2: Characteristic Polynomial: A technique for converting recurrence relations to a closed-form formula.

The characteristic polynomial is a technique used for solving recursively-defined sequences. Recurrence relations are useful when trying to find patterns in number sequences. Remember though, a drawback when using recurrence relations has been given in the example of, "What if I want to find the 100th term in the sequence?" In order to do that you need to know the 99th term, which means you have to do recursion 99 times.

The characteristic polynomial is the first technique you will learn to find what is called a "closed-form formula" or an "explicit formula" for a recurrence relation. Creating this will allow you to easily find the 100th term of the sequence as well as any other term in the sequence.

Example: Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = -3$, $a_1 = -2$ an $n \geq 2$ (Goodaire & Parmenter, 2006, p.181).

You can think of the recurrence relation in terms of a quadratic expression of the form $ax^2 + bx + c$ where a, b and c are constants (numbers). In other words a_n is the ax^2 term, $5a_{n-1}$ is the bx term and $6a_{n-1}$ is the c term.

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$-5a_{n-1} + 6a_{n-2}$ $\quad$ $-5a_{n-1} + 6a_{n-2}$ (Subtract from both sides).

$a_n - 5a_{n-1} + 6a_{n-2} = 0$ (Which converts to) $x^2 - 5x + 6 = 0$.

$(x-2)(x-3) = 0$ (Which gives us roots of) $x_1 = 2$ and $x_2 = 3$

(Next, we put these roots into the equation) $a_n = c_1(x_1{}^n) + c_2(x_2{}^n)$ (Goodaire & Parmenter, 2006, p.171).
$a_n = c_1(2^n) + c_2(3^n)$.

With the first initial condition of $a_0 = -3$ we can substitute 0 for n.

$a_n = c_1(2^n) + c_2(3^n)$ (Becomes) $a_0 = c_1(2^0) + c_2(3^0)$ (Which then becomes) $-3 = c_1 + c_2$ (Remember anything to the zero power is one.)

The second initial condition is $a_1 = -2$. We can substitute 1 for n.
$a_n = c_1(2^n) + c_2(3^n)$ (Becomes) $a_1 = c_1(2^1) + c_2(3^1)$ (Which then becomes) $-2 = 2c_1 + 3c_2$

(We now have two equations with two variables).

$-3 = c_1 + c_2$
$-2 = 2c_1 + 3c_2$

(Solve using elimination or substitution we get).

$c_1 = -7$ And $c_2 = 4$

(Substituting) $c_1 = -7$ and $c_2 = 4$ (into) $a_n = c_1(2^n) + c_2(3^n)$ (yields)

$a_n = -7(2^n) + 4(3^n)$ (Which is the closed-form formula or our solution).

If you test the formula you will see $a_0 = -3$, $a_1 = -2$ and the 100th term is 2061510083000000000000000000000000000000000000000000!

Exercises: For each recurrence relation find the closed-form formula. After you create your formula make sure and test the initial conditions to see if it works.

1) $a_n = 4a_{n-1}$ Given $a_0 = 1$, $a_1 = 4$ where $n \geq 2$

Solution:

$a_n - 4a_{n-1} = 0$
$x^2 - 4x = 0$
$x(x-4) = 0$
$x_1 = 0, x_2 = 4$
$a_n = c_1({x_1}^n) + c_2({x_2}^n)$
$a_0 = 1, a_1 = 4$
$a_0 = c_1(0^0) + c_2(4^0)$
$1 = c_2$
$\therefore a_n = 4^n, n \geq 0$

2) $a_n = -2a_{n-1} + 15a_{n-2}$ Given $a_0 = 1$, $a_1 = 1$ where $n \geq 2$ (Goodaire & Parmenter, 2006, p.174).

Solution:

$a_n + 2a_{n-1} - 15a_{n-2} = 0$

$x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

$x_1 = -5, x_2 = 3$

$a_n = c_1(x_1{}^n) + c_2(x_2{}^n)$

$a_0 \rightarrow 1 = c_1(-5^0) + c_2(3^0)$

$a_1 \rightarrow -1 = c_1(-5^1) + c_2(3^1)$

$1 = c_1 + c_2$

$-1 = -5c_1 + 3c_2$

$c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$

$a_n = \frac{1}{2}(-5^n) + \frac{1}{2}(3^n) \rightarrow$

$\therefore a_n = \frac{1}{2}\left[\left(-5^n\right) + \left(3^n\right)\right] n \geq 0$

3) $a_n = -6a_{n-1} + 7a_{n-2}$ Given $a_0 = 32$, $a_1 = -17$ (Goodaire & Parmenter, 2006, p.174).

$a_n + 6a_{n-1} - 7a_{n-2} = 0$

$x^2 + 6x - 7 = 0$

$(x+7)(x-1) = 0$

$x_1 = -7, x_2 = 1$

$a_n = c_1(x_1{}^n) + c_2(x_2{}^n)$

$a_0 \rightarrow 32 = c_1(-7^0) + c_2(1^0)$

$a_1 \rightarrow -17 = c_1(-7^1) + c_2(1^1)$

$32 = c_1 + c_2$

$-17 = -7c_1 + c_2$

$c_1 = \frac{49}{8} c_2 = \frac{207}{8}$

$\therefore a_n = \frac{49}{8}(-7^n) + \frac{207}{8}, n \geq 0$

4) $a_n = -8a_{n-1} - a_{n-2}$ Given $a_0 = 0$ and $a_1 = 1$ (Goodaire & Parmenter, 2006, p.174).

Solution:

$$a_n + 8a_{n-1} + a_{n-2} = 0$$
$$x^2 + 8x + 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-8 \pm \sqrt{(-8)^2 - 4(1)(1)}}{2(1)}$$
$$x = (-4 \pm \sqrt{15})$$
$$x_1 = (-4 + \sqrt{15}), x_2 = (-4 - \sqrt{15})$$
$$a_n = c_1(x_1{}^n) + c_2(x_2{}^n)$$
$$a_0 \to 0 = c_1((-4 + \sqrt{15})^0) + c_2((-4 - \sqrt{15})^0)$$
$$a_1 \to 1 = c_1((-4 + \sqrt{15})^1) + c_2((-4 - \sqrt{15})^1)$$
$$0 = c_1 + c_2$$
$$1 = c_1(-4 + \sqrt{15}) + c_2(-4 - \sqrt{15})$$
$$c_1 = \frac{\sqrt{15}}{30} c_2 = -\frac{\sqrt{15}}{30}$$
$$a_n = \frac{\sqrt{15}}{30}(-4 + \sqrt{15})^n - \frac{\sqrt{15}}{30}(-4 - \sqrt{15})^n \to$$
$$\therefore a_n = \frac{\sqrt{15}}{30}\left[\left(-4 + \sqrt{15}\right)^n - \left(-4 - \sqrt{15}\right)^n\right], n \geq 0$$

5) $a_{n+1} = 7a_n - 10a_{n-1}$ Given $a_1 = 10$ and $a_2 = 29$ (Goodaire & Parmenter, 2006, p.174).

Solution:

$a_{n+1} = 7a_n - 10a_{n-1}$

$a_{n+1} - 7a_n + 10a_{n-1} = 0$

$x^2 - 7x + 10 = 0$

$(x-5)(x-2) = 0$

$x_1 = 5, x_2 = 2$

$a_n = c_1(x_1)^n + c_2(x_2)^n$

$a_1 \to 10 = c_1(5)^1 + c_2(2)^1$

$a_2 \to 29 = c_1(5)_2 + c_2(2)^1$

$10 = 5c_1 + 2c_2$

$29 = 25c_1 + 4c_2$

$c_1 = \frac{3}{5}, c_2 = \frac{21}{6}$

$\therefore a_n = \frac{3}{5}(5)^n + \frac{21}{6}(2)^n, n \geq 1$

5.3.4: Lesson Reflection

Lesson 2: The Characteristic Polynomial

In lesson 2 I had to do a lot more teaching or modeling than in lesson 1 since understandably they had not seen the characteristic polynomial before. Over the last few weeks we as a class have talked about the difference between a closed form and an explicit formula versus a recurrence relation. The recurrence relation being an equation in which the current term is defined or found from the previous term(s); Whereas the explicit formula will help find an nth term regardless of the previous term(s). The students where very excited and curious to learn a technique that would help them find an explicit formula.

I had to begin by going over the example that I gave on the handout. What helped many of the students was how the characteristic polynomial can be thought of in terms of a quadratic expression. Earlier in the semester we had done a whole unit on quadratics and were familiar with factoring a trinomial into a product of two binomials, solving for the roots, using the quadratic equation when it is not factorable, substitution, and elimination. Some students did have some questions in relation to algebra. Some items that did take some extra explanation included equating a recurrence relation to a quadratic expression symbolically. Another area where some students had confusion was with the concept of the general solution. This was mainly due to notation. For example if we have $a_n = c_1(2^n) + c_2(3^n)$

With the first initial condition of $a_0 = -3$ we can substitute 0 for n

$a_n = c_1(2^n) + c_2(3^n)$ (Becomes) $a_0 = c_1(2^0) + c_2(3^0)$ (Which then becomes) $-3 = c_1 + c_2$.

Some of the problems include, instead of putting the term “a sub-zero” where “a sub-zero” is, some students put 0 in that spot. Also, some students still struggle with that anything to the power of zero is one, some initially think it is zero. After teaching the example I had the students start with problem 1 which was short and simple. Some students’ were confused with how to factor problem 1 since there is no c term and some were confused how the first c term would go to zero when the first x term equals zero. A few of my advanced students did all of 1-5 whereas other students had varying degrees of success. A couple of students almost solved problem 4. One student in particular wanted to know the solution before she went home because she wanted to try the problem again with the help of her dad who is a good tutor in math. This lesson took two periods with some going over again during the second period.

5.4: Guess and Check with the Principle of Mathematical Induction

5.4.1: Lesson Plan

Part 1: Checking the Explicit Formula

Lesson 3: Guess and Check with the Principle of Mathematical Induction

Grade level: High School (9-12)	Subject: Advanced Algebra II	Unit Title: Techniques for creating explicit formulas from recurrence relations	Unit topic: Discrete Math-Recurrence Relations
Lesson# Three-parts 1 and 2	Lesson Title: Guess and Check-part 1 Principle of Mathematical Induction-part 2	Materials: Handout Graphing calculator	Overview and purpose: Check the explicit formula from the characteristic polynomial algebraically. Use induction to check if the explicit formula will work for all $n \geq 0$

Objectives:
The objectives in this lesson are twofold. First students will be able to check their explicit formula obtained using the characteristic polynomial by substituting it in to the recurrence relation. Second, students will learn the technique of guess and check with induction to ensure their explicit formula works for all $n \geq 0$
Information: Part 1: An example is shown on how to check an explicit formula using the recurrence relation. The teacher may go over this example again or a different one. Students will take their explicit formula and substitute it back into the recurrence relation. If the explicit formula is correct and the algebra was done correctly then both sides of the equation will equal zero. Part 2: An example is shown on how to use the principle of mathematical induction to ensure that the explicit formula works for all $n \geq 0$.

In both parts of the lesson students will work through the handout in the context of an investigative tutorial. Conceptually, students should understand that just because you have an explicit formula does not mean that it will work for every *nth* term. The principle of mathematical induction will prove that your explicit formula works for all $n \geq 0$.
Verification: Students will check each other for understanding during the class period by having their peers present their findings in front of the classroom. They will teach each other. Teacher will also check for understanding by prompting and promoting discussion and with open-ended questioning. Either at the end of the lesson or the following class period the teacher will collect student work and grade and give appropriate feedback. The following class period the class should review the previous days lesson to ensure comprehension and understanding before moving on to the next lesson.
Activity: With the handout provided students will go through the handout in a tutorial investigative like approach to complete and comprehend the material. In part 1, students should check to see if their explicit formula they obtained from lesson 2 is correct by substituting it back into the recurrence relation and obtaining a result of zero on both sides of the recurrence relation. In part two, students will use the principle of mathematical induction to ensure that their formula will work for all $n \geq 0$.
Summary: At the end of this lesson, students will have a good understanding of checking the explicit formula for accuracy as well as the technique of the principle of mathematical induction. Students should also understand the connection between recurrence relations, the explicit formula and induction.

5.4.2: Student Handout

Name______________________________
Period___________

Lesson 3: Part 1- Checking the Explicit Formula:

In lesson 2 we used the characteristic polynomial to find an explicit formulas for recurrence relations. We will now check the explicit formulas with the recurrence relations to see if it will work for any of the natural numbers from $[1, \infty)$.

Example: (Checking the Explicit Formula): We have already found from example 1 lesson 2 that the explicit formula for $a_n = 5a_{n-1} - 6a_{n-2}$ where $a_0 = -3, a_1 = -2$, $n \geq 2$, (Goodaire & Parmenter, 2006, p.174), is $a_n = -7(2^n) + 4(3^n)$.

This first example is a check to see if the explicit formula will actually work as defined from the recurrence relation.

(First, substitute)

$a_n = -7(2^n) + 4(3^n)$ (into the recurrence relation below)

$a_n = 5a_{n-1} - 6a_{n-2}$

$-7(2^n) + 4(3^n) = 5a_{n-1} - 6a_{n-2}$

(Below, are the corresponding recurrence relations.)

$$a_n = -7(2^n) + 4(3^n)$$
$$a_{n-1} = -7(2^{n-1}) + 4(3^{n-1})$$
$$a_{n-2} = -7(2^{n-2}) + 4(3^{n-2})$$

(Next, substitute the corresponding recurrence relations into the original.)

$$-7(2^n) + 4(3^n) = 5\left[-7(2^{n-1}) + 4(3^{n-1})\right] - 6\left[-7(2^{n-2}) + 4(3^{n-2})\right]$$

(Then, we can get the equation all on the left side.)

$$-7(2^n) + 4(3^n) - 5\left[-7(2^{n-1}) + 4(3^{n-1})\right] + 6\left[-7(2^{n-2}) + 4(3^{n-2})\right] = 0$$

(Distribute through the brackets to obtain.)

$$-7(2^n) + 4(3^n) + 35(2^{n-1}) - 20(3^{n-1}) - 42(2^{n-2}) + 24(3^{n-2}) = 0$$

(Next, collect like terms.)

$$\left[-7(2^n) + 35(2^{n-1}) - 42(2^{n-2})\right] + \left[4(3^n) - 20(3^{n-1}) + 24(3^{n-2})\right] = 0$$

(Factor)

$$2^{n-2}\left[-7(2^2) + 35(2^1) - 42(2^0)\right] + 3^{n-2}\left[4(3^2) - 20(3^1) + 24(3^0)\right] = 0$$

(Using order of operations you will notice that each bracket simplifies to zero.)

$$2^{n-2}[0] + 3^{n-2}[0] = 0$$

$0 = 0$ Check! This verifies that the explicit formula is valid!

Exercises: For #1-5, check each explicit formula by substituting it into the recurrence relation.

1) Recurrence relation $a_n = 4a_{n-1}$ given $a_0 = 1,\ a_1 = 4$, where $n \geq 1$
   Explicit formula: $a_n = 4^n$ For $n \geq 0$

2) $a_n = -2a_{n-1} + 15a_{n-2}$ Given $a_0 = 1, a_1 = 1$ where $n \geq 2$ (Goodaire & Parmenter, 2006, p.174),

$a_n = \frac{1}{2}\left[\left(-5^n\right) + \left(3^n\right)\right]$ Where $n \geq 0$

3) $a_n = -6a_{n-1} + 7a_{n-2}$ Given, $a_0 = 32, a_1 = -17$ (Goodaire & Parmenter, 2006, p.174),

$a_n = \frac{49}{8}\left(-7^n\right) + \frac{207}{8}$ Where $n \geq 2$

4) $a_n = -8a_{n-1} - a_{n-2}$ Given $a_0 = 0$ and $a_1 = 1$ (Goodaire & Parmenter, 2006,p.174),

$$a_n = \frac{1}{2\sqrt{15}}\left[\left(-4+\sqrt{15}\right) - \left(-4-\sqrt{15}\right)\right] \forall n \geq 0$$

5) $a_{n+1} = 7a_n - 10a_{n-2}$ Given $a_0 = 10$ and $a_1 = 29$ (Goodaire & Parmenter, 2006, p.174),

$$a_n = \frac{3}{5}\left(5^n\right) + \frac{21}{6}\left(2^n\right) \forall n \geq 0$$

5.4.3: Instructor Solutions

Name______________________________
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<u>Lesson 3: Part 1- Checking the Explicit Formula</u>:

In lesson 2 we used the characteristic polynomial to find an explicit formulas for recurrence relations. We will now check the explicit formulas with the recurrence relations to see if it will work for any of the natural numbers from $[1, \infty)$.

<u>Example: (Checking the Explicit Formula):</u> We have already found from example 1 lesson 2 that the explicit formula for $a_n = 5a_{n-1} - 6a_{n-2}$ where $a_0 = -3, a_1 = -2$, $n \geq 2$, (Goodaire & Parmenter, 2006, p.181), is $a_n = -7(2^n) + 4(3^n)$.

This first example is a check to see if the explicit formula will actually work as defined from the recurrence relation.

(First, substitute)

$a_n = -7(2^n) + 4(3^n)$ (into the recurrence relation below)

$a_n = 5a_{n-1} - 6a_{n-2}$

$-7(2^n) + 4(3^n) = 5a_{n-1} - 6a_{n-2}$

(Below, are the corresponding recurrence relations.)

$$a_n = -7(2^n) + 4(3^n)$$
$$a_{n-1} = -7(2^{n-1}) + 4(3^{n-1})$$
$$a_{n-2} = -7(2^{n-2}) + 4(3^{n-2})$$

(Next, substitute the corresponding recurrence relations into the original.)

$$-7(2^n) + 4(3^n) = 5\left[-7(2^{n-1}) + 4(3^{n-1})\right] - 6\left[-7(2^{n-2}) + 4(3^{n-2})\right]$$

(Then, we can get the equation all on the left side.)

$$-7(2^n) + 4(3^n) - 5\left[-7(2^{n-1}) + 4(3^{n-1})\right] + 6\left[-7(2^{n-2}) + 4(3^{n-2})\right] = 0$$

(Distribute through the brackets to obtain.)

$-7(2^n) + 4(3^n) + 35(2^{n-1}) - 20(3^{n-1}) - 42(2^{n-2}) + 24(3^{n-2}) = 0$

(Next, collect like terms.)

$\left[-7(2^n) + 35(2^{n-1}) - 42(2^{n-2})\right] + \left[4(3^n) - 20(3^{n-1}) + 24(3^{n-2})\right] = 0$

(Factor)

$2^{n-2}\left[-7(2^2) + 35(2^1) - 42(2^0)\right] + 3^{n-2}\left[4(3^2) - 20(3^1) + 24(3^0)\right] = 0$

(Using order of operations you will notice that each bracket simplifies to zero.)

$2^{n-2}[0] + 3^{n-2}[0] = 0$

$0 = 0$ Check! This verifies that the explicit formula is valid!

<u>**Exercises:**</u> For #1-5, check each explicit formula by substituting it into the recurrence relation.

1) Recurrence relation $a_n = 4a_{n-1}$ given $a_0 = 1,\ a_1 = 4$, where $n \geq 1$
Explicit formula: $a_n = 4^n$ For $n \geq 0$

Solution:

$4^n = 4a_{n-1}$
$4^n = 4(4^{n-1})$
$4^n - 4(4^{n-1}) = 0$
$4^{n-1}(4^1 - 4) = 0$
$4^{n-1}(0) = 0$
$0 = 0$

2) $a_n = -2a_{n-1} + 15a_{n-2}$ Given $a_0 = 1, a_1 = 1$ where $n \geq 2$ (Goodaire & Parmenter, 2006, p.174),

$a_n = \frac{1}{2}\left[\left(-5^n\right) + \left(3^n\right)\right]$ Where $n \geq 0$

Solution:

$$\frac{1}{2}(-5^n) + \frac{1}{2}(3^n) = -2\left[\frac{1}{2}(-5^{n-1}) + \frac{1}{2}(3^{n-1})\right] + 15\left[\frac{1}{2}(-5^{n-2}) + \frac{1}{2}(3^{n-2})\right]$$

$$\frac{1}{2}(-5^n) + \frac{1}{2}(3^n) = -2\left[\frac{1}{2}(-5^{n-1}) + \frac{1}{2}(3^{n-1})\right] + 15\left[\frac{1}{2}(-5^{n-2}) + \frac{1}{2}(3^{n-2})\right]$$

$$\frac{1}{2}(-5^n) + \frac{1}{2}(3^n) + 2\left[\frac{1}{2}(-5^{n-1}) + \frac{1}{2}(3^{n-1})\right] - 15\left[\frac{1}{2}(-5^{n-2}) + \frac{1}{2}(3^{n-2})\right] = 0$$

$$\frac{1}{2}(-5^n) + \frac{1}{2}(3^n) + (-5^{n-1}) + (3^{n-1}) - \frac{15}{2}(-5^{n-2}) - \frac{15}{2}(3^{n-2}) = 0$$

$$\frac{1}{2}(-5^n) + (-5^{n-1}) - \frac{15}{2}(-5^{n-2}) + \frac{1}{2}(3^n) + (3^{n-1}) - \frac{15}{2}(3^{n-2}) = 0$$

$$-5^{n-2}\{\frac{1}{2}(-5^2) + (-5^1) - \frac{15}{2}(-5^0)\} + 3^{n-2}\{\frac{1}{2}(3^2) + (3^1) - \frac{15}{2}(3^0)\} = 0$$

$$-5^{n-2}\{0\} + 3^{n-2}\{0\} = 0$$

$$0 + 0 = 0$$

$$0 = 0$$

3) $a_n = -6a_{n-1} + 7a_{n-2}$ Given, $a_0 = 32, a_1 = -17$ (Goodaire & Parmenter, 2006, p.174),

$a_n = \frac{49}{8}\left(-7^n\right) + \frac{207}{8}$ Where $n \geq 2$

Solution:

$$\frac{49}{8}(-7^n) + \frac{207}{8} = -6\left[\frac{49}{8}(-7^{n-1}) + \frac{207}{8}\right] + 7\left[\frac{49}{8}(-7^{n-2}) + \frac{207}{8}\right]$$

$$\frac{49}{8}(-7^n) + \frac{207}{8} + 6\left[\frac{49}{8}(-7^{n-1}) + \frac{207}{8}\right] - 7\left[\frac{49}{8}(-7^{n-2}) + \frac{207}{8}\right] = 0$$

$$\frac{49}{8}(-7^n) + \frac{207}{8} + \frac{294}{8}(-7^{n-1}) + \frac{1242}{8} - \frac{343}{8}(-7^{n-2}) - \frac{1449}{8} = 0$$

$$-7^{n-2}\{\frac{49}{8}(-7^2) + \frac{207}{8} + \frac{294}{8}(-7^1) + \frac{1242}{8} - \frac{343}{8}(-7^0) - \frac{1449}{8}\} = 0$$

$$-7^{n-2}\{0\} = 0$$

$$0 = 0$$

4) $a_n = -8a_{n-1} - a_{n-2}$ Given $a_0 = 0$ and $a_1 = 1$ (Goodaire & Parmenter, 2006, p.174),

$$a_n = \frac{1}{2\sqrt{15}}\left[\left(-4+\sqrt{15}\right)-\left(-4-\sqrt{15}\right)\right]\forall n \geq 0$$

Solution:

$$\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^n - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^n = -8\left[\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^{n-1} - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^{n-1}\right] -$$

$$\left[\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^{n-2} - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^{n-2}\right]$$

$$\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^n - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^n + 8\left[\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^{n-1} - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^{n-1}\right] +$$

$$\left[\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^{n-2} - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^{n-2}\right] = 0$$

$$\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^n - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^n + \frac{4}{\sqrt{15}}(-4+\sqrt{15})^{n-1} - \frac{4}{\sqrt{15}}(-4-\sqrt{15})^{n-1} + \frac{1}{2\sqrt{15}}(-4+\sqrt{15})^{n-2} -$$

$$\frac{1}{2\sqrt{15}}(-4-\sqrt{15})^{n-2} = 0$$

$$\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^n + \frac{4}{\sqrt{15}}(-4+\sqrt{15})^{n-1} + \frac{1}{2\sqrt{15}}(-4+\sqrt{15})^{n-2} - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^n - \frac{4}{\sqrt{15}}(-4-\sqrt{15})^{n-1} -$$

$$\frac{1}{2\sqrt{15}}(-4-\sqrt{15})^{n-2} = 0$$

$$(-4+\sqrt{15})^{n-2}\{\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^2 + \frac{4}{\sqrt{15}}(-4+\sqrt{15})^1 + \frac{1}{2\sqrt{15}}(-4+\sqrt{15})^0\} + (-4-\sqrt{15})^{n-2}\{-\frac{1}{2\sqrt{15}}(-4-\sqrt{15})^2 -$$

$$\frac{4}{\sqrt{15}}(-4-\sqrt{15})^1 - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^0\} = 0$$

$$(-4+\sqrt{15})^{n-2}\{0\} + (-4-\sqrt{15})^{n-2}\{0\} = 0$$

$$0 + 0 = 0$$

$$0 = 0$$

5) $a_{n+1} = 7a_n - 10a_{n-2}$ Given $a_0 = 10$ and $a_1 = 29$ (Goodaire & Parmenter, 2006, p.174), $a_n = \frac{3}{5}\left(5^n\right)+\frac{21}{6}\left(2^n\right)\forall n \geq 0$

Solution:

$$\frac{3}{5}(5^{n+1})+\frac{21}{6}(2^{n+1}) = 7\left[\frac{3}{5}(5^n)+\frac{21}{6}(2^n)\right]-10\left[\frac{3}{5}(5^{n-1})+\frac{21}{6}(2^{n-1})\right]$$
$$\frac{3}{5}(5^{n+1})+\frac{21}{6}(2^{n+1})-7\left[\frac{3}{5}(5^n)+\frac{21}{6}(2^n)\right]+10\left[\frac{3}{5}(5^{n-1})+\frac{21}{6}(2^{n-1})\right]=0$$
$$\frac{3}{5}(5^{n+1})+\frac{21}{6}(2^{n+1})-\frac{21}{5}(5^n)-\frac{147}{6}(2^n)+6(5^{n-1})+\frac{210}{6}(2^{n-1})=0$$
$$\frac{3}{5}(5^{n+1})-\frac{21}{5}(5^n)+6(5^{n-1})+\frac{21}{6}(2^{n+1})-\frac{147}{6}(2^n)+\frac{210}{6}(2^{n-1})=0$$
$$5^{n-1}\{\frac{3}{5}(5^2)-\frac{21}{5}(5^1)+6(5^0)\}+2^{n-1}\{\frac{21}{6}(2^2)-\frac{147}{6}(2^1)+\frac{210}{6}(2^0)\}=0$$
$$5^{n-1}[0]+2^{n-1}[0]=0$$
$$0+0=0$$
$$0=0$$

5.4.4: Student Handout

Part 2: Guess and Check with Induction

Name______________________
Period__________

Lesson 3: Part 2-Guess and Check with Induction:

Induction is a technique to prove that recurrence relations that are converted to a closed form formula will work for any of the natural numbers $[1, \infty]$.
In lesson 2 you learned a technique for converting recurrence relations to closed form formulas known as the characteristic polynomial. From the example in lesson 2 your were shown how to solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = -3$, $a_1 = -2$, for $n \geq 2$ we get a closed-form formula of $a_n = -7(2^n) + 4(3^n)$.

After that you tested the initial conditions and maybe several other terms to see if the closed form formula works. The problem is, how do you know that your formula will work for all of the natural numbers? The Principle of Mathematical Induction proves that your formula will work for all the natural numbers. (The natural numbers (also known as the counting numbers) go from $[1, \infty]$).

The Principle of Mathematical Induction states:

If P_n is some statement about some natural number n and
a) P_1 Is true (base case), and
b) Assuming P_k is true implies that P_{k+1} is true,
Then P_n must be true for all positive integers n. (Smith, Charles, Dossey, Bittenger, 2001, p.634).

Example: Induction

Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1 = 1$, and $a_{k+1} = 3a_k$ for $k \geq 1$. Prove that $a_n = 3^{n-1}$ for all $n \geq 1$. (Goodaire & Parmenter, 2006, p.167).

Proof:
1) Base case for $n = 1$, $a_1 = 3^{1-1} = 3^0 = 1$. Check
2) Assume $a_n = 3^{n-1}$ is true, then w.t.s. $a_{n+1} = 3^{(n+1)-1} = 3^n$ Implies that $a_n = 3^n$ is True. Then consider $a_{k+1} = 3a_k$ which is given, then $a_{n+1} = 3a_n = 3^1(3^{n-1}) = 3^n$. Check. Therefore by induction (1 and 2) $a_n = 3^{n-1}$ is true for all $n \geq 1$.

Exercises: For exercises #1-4 use induction to prove the explicit formula is correct. For problem 4 find the first 6 terms of the sequence, guess a formula for the recurrence relation and use induction to prove your formula is correct.

1) Prove:

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

(Smith et al., 2001, p.635).

2) Prove that the sum of n consecutive positive odd integers is n^2. In other words prove that $1+3+5+\ldots+(2n-1)=n^2$ (Smith et al., 2001, p.634).

3) Suppose $a_1, a_2, a_3, \ldots$ is a sequence of integers such that $a_1 = 0$ and for $n \geq 1$, $a_n = n^3 + a_{n-1}$. Prove that $a_n = \frac{(n-1)(n-2)\left(n^2+n+2\right)}{4}$ for every integer $n \geq 1$. (Goodaire & Parmenter, 2006, p.167).

4) Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1 = 1$ and for $n \geq 1$, $a_n = 2a_{n-1} + 1$. Write down the first six terms of the sequence. Guess a formula for a_n and prove that your guess is correct (Goodaire & Parmenter, 2006, p.167).

5.4.5: Instructor Solutions

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Lesson 3: Part 2-Guess and Check with Induction

Induction is a technique to prove that recurrence relations that are converted to a closed form formula will work for any of the natural numbers $[1, \infty]$.
In lesson 2 you learned a technique for converting recurrence relations to closed form formulas known as the characteristic polynomial. From the example in lesson 2 your were shown how to solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = -3$, $a_1 = -2$, for $n \geq 2$ we get a closed-form formula of $a_n = -7(2^n) + 4(3^n)$.

After that you tested the initial conditions and maybe several other terms to see if the closed form formula works. The problem is, how do you know that your formula will work for all of the natural numbers? The Principle of Mathematical Induction proves that your formula will work for all the natural numbers. (The natural numbers (also known as the counting numbers) go from $[1, \infty]$).

The Principle of Mathematical Induction states:

If P_n is some statement about some natural number n and
a) P_1 Is true (base case), and
b) Assuming P_k is true implies that P_{k+1} is true,
Then P_n must be true for all positive integers n. (Smith et al., 2001, p.634).

Example: Induction

Let $a_1, a_2, a_3, \ldots$ be the sequence defined by $a_1 = 1$, and $a_{k+1} = 3a_k$ for $k \geq 1$. Prove that $a_n = 3^{n-1}$ for all $n \geq 1$. (Goodaire & Parmenter, 2006, p.167).

Proof:

1) Base case for $n = 1$, $a_1 = 3^{1-1} = 3^0 = 1$. Check
2) Assume $a_n = 3^{n-1}$ is true, then w.t.s. $a_{n+1} = 3^{(n+1)-1} = 3^n$ Implies that $a_n = 3^n$ is True. Then consider $a_{k+1} = 3a_k$ which is given, then $a_{n+1} = 3a_n = 3^1(3^{n-1}) = 3^n$. Check. Therefore by induction (1 and 2) $a_n = 3^{n-1}$ is true for all $n \geq 1$.

Exercises: For exercises #1-4 use induction to prove the explicit formula is correct. For problem 4 find the first 6 terms of the sequence, guess a formula for the recurrence relation and use induction to prove your formula is correct

1) Prove:
$1+2+3+\ldots+n=\frac{n(n+1)}{2}$ (Smith et al., 2001, p.635).

Solution:

a) Base case when $n=1 \to 1+2+3+\ldots+1=\frac{1(1+1)}{2} \to 1=1$. Check
b) Proof: Suppose $n=k$ that is $1+2+3+\ldots+k=\frac{k(k+1)}{2}$, w.t.s. That the statement is true when $n=k+1$. Adding $k+1$ to both sides results in:

$$1+2+3+\ldots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$
$$=\frac{k(k+1)+2(k+1)}{2} \to \frac{k^2+k+2k+2}{2} \to$$
$$\frac{k^2+3k+2}{2} \to \frac{(k+1)(k+2)}{2}$$

$\therefore$ By induction, $a_n=\frac{n(n+1)}{2}$ For all $n \geq 1$.

2) Prove that the sum of n consecutive positive odd integers is n^2. In other words prove that $1+3+5+\ldots+(2n-1)=n^2$ (Smith et al., 2001, p.635).

Solution: (Smith et. al, 2001, p.634-635).

a) Base case for when $n=1 \to (2 \bullet 1-1)=1^2 \to 1=1$. Check
b) Proof: Suppose $n=k$, that is, $1+3+5+\ldots+(2k-1)=k^2$, w.t.s. That the statement is true when $n=k+1$. $1+3+5+\ldots+(2k-1)=k^2$ Assumed true for k. Adding $2(k+1)-1$ to both sides results in:

$$1+3+5+\ldots+(2k-1)+[2(k+1)-1]=k^2+[2(k+1)-1]$$
$$=k^2+2k+2-1 \to k^2+2k+1 \to (k+1)^2$$

$\therefore$ By induction

$a_n=n^2$ Is true for all $n \geq 1$.

3) Suppose $a_1, a_2, a_3, \dots$ is a sequence of integers such that $a_1 = 0$ and for $n \geq 1$, $a_n = n^3 + a_{n-1}$. Prove that $a_n = \frac{(n-1)(n-2)(n^2+n+2)}{4}$ for every integer $n \geq 1$. (Goodaire & Parmenter, 2006, p.167).

Solution: (Goodaire & Parmenter, 2006, p. S-28).

a) Base case for when $n = 1 \rightarrow a_1 = \frac{(1-1)(1+2)(1^2+1+2)}{4} \rightarrow a_1 = 0$. Check

b) Proof: Suppose $n = k$ and assume $a_k = \frac{(k-1)(k+2)(k^2+k+2)}{4}$ is true, then

$$a_{k+1} = \frac{((k+1)-1)((k+1)+2)((k+1)^2+(k+1)+2)}{4} \text{ is true}$$

Consider $a_n = n^3 + a_{n-1}$ which is given then $a_k = k^3 + a_{k-1}$ results in:

$$a_{k+1} = (k+1)^3 + a_{k+1-1}$$
$$= (k+1)^3 + a_k$$
$$= (k+1)^3 + \frac{(k-1)(k+2)(k^2+k-2)}{4}$$
$$= \frac{4(k^2+2k+1)(k+1) + (k^2+k-2)(k^2+k+2)}{4}$$
$$= \frac{k^4+6k^3+13k^2+12k}{4}$$
$$= \frac{(k)(k+3)((k+1)^2+k+3)}{4}$$
$$\therefore \quad a_n = \frac{(n-1)(n+2)(n^2+n+2)}{4} \quad \text{For all } n \geq 1.$$

4) Let $a_1, a_2, a_3, \dots$ be the sequence defined by $a_1 = 1$ and for $n \geq 1$, $a_n = 2a_{n-1} + 1$. Write down the first six terms of the sequence. Guess a formula for a_n and prove that your guess is correct (Goodaire & Parmenter, 2006, p.167).

Solution: (Goodaire & Parmenter, 2006, p. S-28).

$a_1 = 1, a_2 = 3, a_3 = 7, a_4 = 15, a_5 = 31, a_6 = 63$

a) Base case when $n = 1 \rightarrow 1 + 2 + 3 + \dots + 1 = \frac{1(1+1)}{2} \rightarrow 1 = 1$. Check

b) Proof: Suppose $n = k$ that is $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$, w.t.s. That the statement is true when $n = k+1$. Adding $k+1$ to both sides results in:

$$1+2+3+\ldots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

$$=\frac{k(k+1)+2(k+1)}{2} \rightarrow \frac{k^2+k+2k+2}{2} \rightarrow$$

$$\frac{k^2+3k+2}{2} \rightarrow \frac{(k+1)(k+2)}{2}$$

$\therefore$ By induction, $a_n = \frac{n(n+1)}{2}$ holds for all $n \geq 1$.

5.4.6: Lesson Reflection

<u>Lesson 3: Parts 1 and 2-Checking the Explicit Formula and the Principle of Mathematical Induction</u>

Part 1 of lesson 3 checking the solution for accuracy was fairly successful. Since students have had Algebra 1 and Algebra 2 they were familiar with checking solutions by substituting in the solution to the original equation(s) to do a check. Most of the parts of the process of this lesson were familiar except for when the exponent on some of the terms was $n-1$ and $n-2$ and n. Students had trouble with rules of exponents and also factoring with those same exponents. So other than algebraic problems most students felt comfortable with the process they were doing.

Part 2 of this lesson by far was the least successful and the hardest for the students. It did not help that about two-thirds of the class was gone because of a school activity. Students that were there had trouble grasping the topic conceptually. They had a hard time with notation, and with part 2 in the principle of mathematical induction. We only got through 2 problems. Most of the period was taken up with going over the example that I modeled for them. They had many questions. In the end maybe if I look at it in a different way it was not that it was unsuccessful it was that it was more challenging for them. Even if the students took away from the lesson some new insight and mathematical ideas then it was worth it.

5.5: The Pell Sequence

5.5.1: Lesson Plan

Lesson 4: The Pell Sequence

Grade level:	Subject:	Unit Title:	Unit topic:
High School (9-12)	Advanced Algebra II	Techniques for creating explicit formulas from recurrence relations	Discrete Math-Recurrence Relations
Lesson# Four	Lesson Title: The Pell Sequence	Materials: Handout Graphing calculator	Overview and purpose: Use the characteristic polynomial to solve the Pell Sequence. Use an alternate explicit formula to find terms of the Pell Sequence.

Objectives:
Now that students are familiar with the characteristic polynomial they will use this technique to solve the Pell Sequence. Next they will use an alternate explicit formula to find the first few terms of the Pell Sequence.
Information: In this lesson students will use the characteristic polynomial to get an explicit formula for the Pell Sequence. This recurrence relation should be a challenge for many of the students since it involves roots that are irrational as well as using the quadratic formula. For the instructor there should be no need to model this problem since the students should have all the tools necessary to solve the first problem. In problem 2 the instructor should give a brief review on factorials and maybe one example to illustrate how to use this unique explicit formula. Problem 2 will show also that there are alternate methods or explicit formulas for the same recurrence relation.

Verification:

Students will check each other for understanding during the class period by having their peers present their findings in front of the classroom. They will teach each other. Teacher will also check for understanding by prompting and promoting discussion and with open-ended questioning. Either at the end of the lesson or the following class period the teacher will collect student work and grade and give appropriate feedback. The following period the class should review the previous days lesson to ensure comprehension and understanding before moving on to the next lesson.

Activity:

With the handout provided students will go through the handout in a tutorial investigative like approach to complete and comprehend the material. The modeling or teaching by the instructor should be minimal compared to other lessons since most of the tools being used the students have already learned.

Summary:

At the end of this lesson students should feel more comfortable and confident in using the characteristic polynomial with a more challenging recurrence relation.

5.5.2: Student Handout

Name______________________________
Period_________

Lesson 4: The Pell Sequence

Exercises:

1) The Pell Sequence is defined by $p_0 = 1$, $p_1 = 2$ and $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$. (Goodaire & Parmenter, 2006, p.182).

a) Find the first 6 terms of the sequence

b) Use the characteristic polynomial technique to solve this recurrence relation (Goodaire & Parmenter, 2006, p.182).

c) There exist closed-form solutions for p_n (Pell Sequence). Below, is one example. Calculate the first 6 terms of the sequence by using the formula.

$$p_n = \sum_{\substack{i,j,k \geq 0 \\ i+j+2k=n}} \frac{(i+j+k)!}{i!\,j!\,k!}$$ (Goodaire & Parmenter, 2006, p.182).

5.5.3: Instructor Solutions

Name______________________________
Period_________

Lesson 4: The Pell Sequence

Exercises:

1) The Pell Sequence is defined by $p_0 = 1$, $p_1 = 2$ and $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$. (Goodaire & Parmenter, 2006, p.182).

a) Find the first 6 terms of the sequence.

Solution:

$1,2,5,12,29,70,169,408,\ldots$

b) Use the characteristic polynomial technique to solve this recurrence relation. (Goodaire & Parmenter, 2006, p.182).

Solution:

$$p_n - 2p_{n-1} - p_{n-2} = 0$$
$$x^2 - 2x - 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$
$$x = (1 \pm \sqrt{2})$$
$$x_1 = (1+\sqrt{2}), x_2 = (1-\sqrt{2})$$
$$p_n = c_1(x_1^{\,n}) + c_2(x_2^{\,n})$$
$$p_0 \to 1 = c_1((1+\sqrt{2})^0) + c_2((1-\sqrt{2})^0)$$
$$p_1 \to 2 = c_1((1+\sqrt{2})^1) + c_2((1-\sqrt{2})^1)$$
$$1 = c_1 + c_2$$
$$2 = c_1(1+\sqrt{2}) + c_2(1-\sqrt{2})$$
$$c_1 = \frac{2+\sqrt{2}}{4} c_2 = \frac{2-\sqrt{2}}{4}$$
$$\therefore p_n = (\frac{2+\sqrt{2}}{4})(1+\sqrt{2})^n + (\frac{2-\sqrt{2}}{4})(1-\sqrt{2})^n, n \geq 0$$

2) There exist closed form solutions for p_n (Pell Sequence). Below is one example. Calculate the first 6 terms of the sequence by using the formula below.

$$p_n = \sum_{\substack{i,j,k\geq 0 \\ i+j+2k=n}} \frac{(i+j+k)!}{i!\,j!\,k!}$$ (Goodaire & Parmenter, 2006, p.182).

Solution:

$$p_0 = 1$$
$$i + j + 2k = n$$
$$i + j + 2k = 0$$
$$(i, j, k)$$
$$(0,0,0)$$
$$\frac{(0+0+0)!}{0!0!0!} = \frac{1}{1} = 1$$

$$p_1 = 2$$
$$i + j + 2k = n$$
$$i + j + 2k = 1$$
$$(i, j, k)$$
$$(0,1,0) + (1,0,0)$$
$$\frac{(0+1+0)!}{0!1!0!} + \frac{(1+0+0)!}{1!0!0!}$$
$$\frac{1}{1} + \frac{1}{1} = 1 + 1 = 2$$

$$p_2 = 5$$
$$i + j + 2k = n$$
$$i + j + 2k = 2$$
$$(i, j, k)$$
$$(2,0,0) + (1,1,0) + (0,2,0) + (0,0,1)$$
$$\frac{(2+0+0)!}{2!0!0!} + \frac{(1+1+0)!}{1!1!0!} + \frac{(0+2+0)!}{0!2!0!} + \frac{(0+0+1)!}{0!0!1!}$$
$$\frac{2}{2} + \frac{2}{1} + \frac{2}{2} + \frac{1}{1} = 1 + 2 + 1 + 1 = 5$$

$$p_3 = 12$$
$$i + j + 2k = n$$
$$i + j + 2k = 3$$
$$(i, j, k)$$
$$(0,1,1) + (1,0,1) + (2,1,0) + (1,2,0) + (3,0,0) + (0,3,0)$$
$$\frac{(0+1+1)!}{0!1!1!} + \frac{(1+0+1)!}{1!0!1!} + \frac{(2+1+0)!}{2!1!0!} + \frac{(1+2+0)!}{1!2!0!} + \frac{(3+0+0)!}{3!0!0!} + \frac{(0+3+0)!}{0!3!0!}$$
$$2+2+3+3+1+1=12$$

$$p_4 = 29$$
$$i + j + 2k = n$$
$$i + j + 2k = 4$$
$$(i, j, k)$$
$$(0,2,1) + (2,0,1) + (1,1,1) + (0,0,2) + (1,3,0) + (3,1,0) + (4,0,0) + (0,4,0) + (2,2,0)$$
$$\frac{(0+2+1)!}{0!2!1!} + \frac{(2+0+1)!}{2!0!1!} + \frac{(1+1+1)!}{1!1!1!} + \frac{(0+0+2)!}{0!0!2!} + \frac{(1+3+0)!}{1!3!0!} + \frac{(3+1+0)!}{3!1!0!} \frac{(4+0+0)!}{4!0!0!} + \frac{(0+4+0)!}{0!4!0!}$$
$$+ \frac{(2+2+0)!}{2!2!0!}$$
$$3+3+6+1+4+4+1+1+6=29$$

5.5.3: Lesson Reflection

Lesson 4: The Pell Sequence

Coming off the hard and complicated lesson of guess and check with induction we next turned our attention to the Pell sequence. The first problem went well of finding the first 6 terms in the sequence. To my surprise more students than I thought got an explicit formula for the Pell Sequence. I had them use the Characteristic Polynomial since they were most familiar and proficient at that. These first two problems were done during the first 50 minutes to an hour of a period. The last 25-30 minutes I went over problem 3 from the AMM article, which was another version of a closed form formula for the Pell sequence. I reviewed factorials and I taught the students summation notation and what all of the symbols mean. I also demonstrated on their graphing calculators how to find and use the factorial operation. I did this by showing them how to compute the first two terms, p_0 and p_1.

Other than learning the different symbols and the notation, the only problem they had were some students were getting confused with the index number n. For example, in the case below some students thought n was 1. These students were trying to find combinations where $i, j,$ and k would sum up to one rather than 0.

$$p_0 = 1$$
$$i + j + 2k = n$$
$$i + j + 2k = 0$$
$$(i, j, k)$$
$$(0,0,0)$$
$$\frac{(0+0+0)!}{0!0!0!} = \frac{1}{1} = 1$$

Like the characteristic polynomial this lesson went well. In fact out of the four lessons so far I felt this one went the best. At this point they have had enough teaching, confidence and experience to handle different recurrence relations. Being comfortable with the characteristic polynomial as well quadratics and the simplicity of factorials made this one of the most successful and fun so far. This took one lesson with questions at the beginning of the next period.

5.6: Tower of Hanoi

5.6.1: Lesson Plan

Lesson 5: The Tower of Hanoi

Grade level:	Subject:	Unit Title:	Unit topic:
High School (9-12)	Advanced Algebra II	Techniques for creating explicit formulas from recurrence relations	Discrete Math-Recurrence Relations
Lesson# Five	Lesson Title: The Tower of Hanoi	Materials: Handout Graphing calculator Class set of the tower of Hanoi	Overview and purpose: Generate a sequence, give a recurrence relation and an explicit formula through the Tower of Hanoi problem

Objectives:

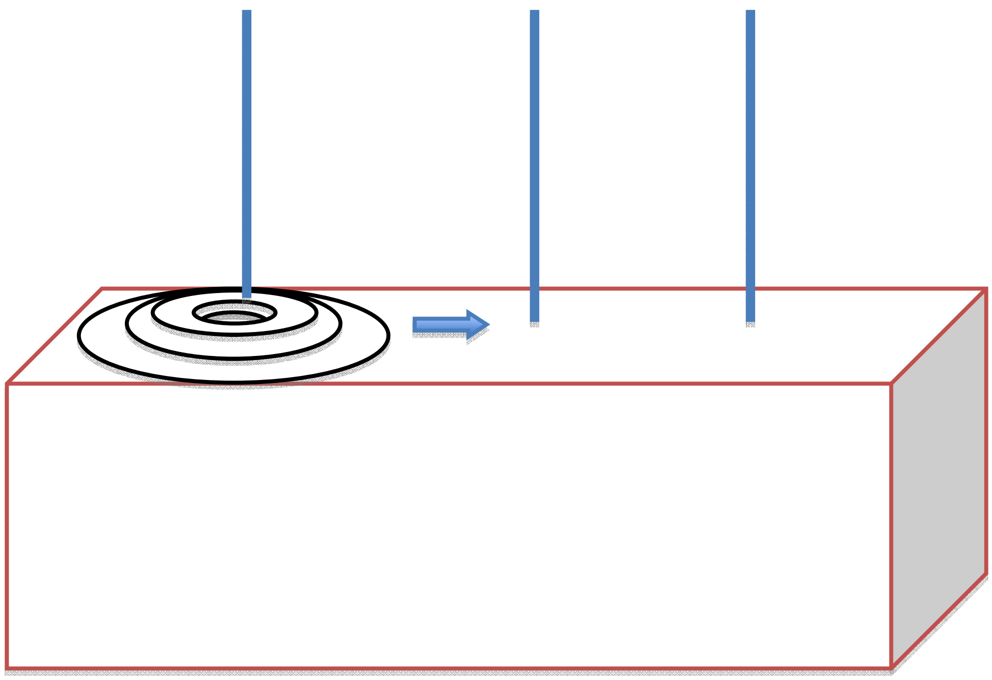
Students will use the Tower of Hanoi problem to generate a sequence, give a recurrence relation and an explicit formula.

Information:

In this lesson it is not necessary, but a class set of The Tower of Hanoi would be helpful. Tower of Hanoi puzzles can be bought off the internet or can be made. I made one using florist foam or Styrofoam for the base, Pencils for the 3 rods and different size O washers for the disks. If the Tower of Hanoi puzzle is not available then the instructor and/or students can use a table to show the moves.
Groups of 2 to 5 work best. Have students make a table for each amount of "n" discs. Students should be able to do $n = 0,1,2,3,4,5$ discs with 6 or more being more challenging as well as time consuming.

This lesson similar to the Pell Sequence should not need much modeling in regards to the activity because students have learned many of the tools necessary to do this activity. A brief demonstration on how the puzzle works will suffice. A quick review on scientific notation might help with the last question on the handout.

Verification: Students will check each other for understanding during the class period by having their peers present their findings in front of the classroom. They will teach each other. Teacher will also check for understanding by prompting and promoting discussion and with open-ended questioning. Either at the end of the lesson or the following class period the teacher will collect student work and grade and give appropriate feedback. The following class period the class should review the previous days lesson to ensure comprehension and understanding before moving on to the next lesson.
Activity: With the handout and the Tower of Hanoi puzzle, students will create tables for $n = 0,1,2,3,4,5$ with 6 or more being optional. After doing this, students will guess and check to find a recurrence relation. They will then create an explicit formula by checking all the terms in relation to their moves. Finally, they will answer a question, which asks how long it will take with different amounts of n moves.
Summary: The Tower of Hanoi puzzle will show the students that a sequence, a recurrence relation and an explicit formula can all be created with the use of a model. This model as well as others may model an application setting in the real world as well as generate discussion as to where recursion can be used in real life settings.



5.6.2: Student Handout

Name______________________________
Period____________

<u>Lesson 5: The Tower of Hanoi</u>

Professor Claus introduced the Tower of Hanoi puzzle in 1883. It consists of three pegs and a number of disks of differing diameters, each with a hole in the center. The discs initially sit on one of the pegs in order of decreasing diameter (smallest at top, largest at bottom), thus forming a triangular tower. The object is to move the tower to one of the other pegs by transferring the discs to any peg one at a time in such a way that no disc is ever placed upon a smaller one (Merris, 2003, p.332-333).

Problem 1 (parts a thru e): (Goodaire & Parmenter, 2006, p.175).

a) Solve the puzzle when there are $n = 0,1,2,3,4,5$ discs and show your moves by completing a table for each disc amount like shown below. [The pegs are labeled A, B, C and use an asterisk (*) to denote an empty peg. The discs are numbered in order of increasing size, thus disc 1 is the smallest.]

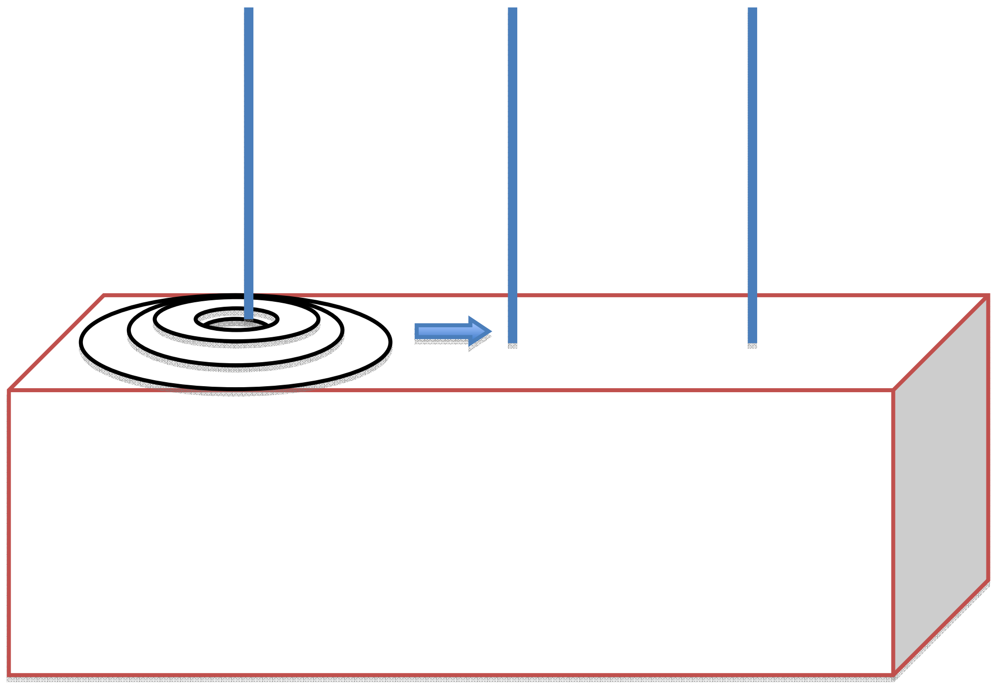
Example of a table below with n=2 discs.

	A	B	C
Initial Position	1,2	*	*
Move 1	??	??	??
Move 2	??	??	??
Etc.			

b) Give a recurrence relation for a_n, the number of moves required to transfer n discs from one peg to another.

c) Find an explicit formula for a_n

d) Suppose we can move a disc a second. Estimate the time required to transfer the discs if $n = 8$, $n = 16$, and $n = 64$.



5.6.3: Instructor Solutions

Name______________________________
Period____________

Lesson 5: The Tower of Hanoi

Professor Claus introduced the Tower of Hanoi puzzle in 1883. It consists of three pegs and a number of disks of differing diameters, each with a hole in the center. The discs initially sit on one of the pegs in order of decreasing diameter (smallest at top, largest at bottom), thus forming a triangular tower. The object is to move the tower to one of the other pegs by transferring the discs to any peg one at a time in such a way that no disc is ever placed upon a smaller one (Merris, 2003, p.332-333).

Problem 1 (parts a thru e): (Goodaire & Parmenter, 2006, p.175).

a) Solve the puzzle when there are $n = 0,1,2,3,4,5$ discs and show your moves by completing a little table for each disc amount like shown below. [The pegs are labeled A, B, C and use an asterisk (*) to denote an empty peg. The discs are numbered in order of increasing size, thus disc 1 is the smallest.]

Solution:

Table with n=0 discs.

	A	B	C
Initial Position and Final Position	*	*	*

Table with n=1 discs.

	A	B	C
Initial Position	1	*	*
Move 1	*	1	*

Table with n=2 discs.

	A	B	C
Initial Position	1,2	*	*
Move 1	2	*	1
Move 2	*	2	1
Move 3	*	1,2	*

Table with n=3 discs.

	A	B	C
Initial Position	1,2,3	*	*
Move 1	2,3	*	1
Move 2	3	2	1
Move 3	3	1,2	*
Move 4	*	1,2	3
Move 5	1	2	3
Move 6	1	*	2,3
Move 7	*	*	1,2,3

Table with n=4 discs.

	A	B	C
Initial Position	1,2,3,4	*	*
Move 1	2,3,4	*	1
Move 2	3,4	2	1
Move 3	3,4	1,2	*
Move 4	4	1,2	3
Move 5	1,4	2	3
Move 6	1,4	*	2,3
Move 7	4	*	1,2,3
Move 8	*	4	1,2,3
Move 9	*	1,4	2,3
Move 10	2	1,4	3
Move 11	1,2	4	3
Move 12	1,2	3,4	*
Move 13	2	3,4	1
Move 14	*	2,3,4	1
Move 15	*	1,2,3,4	*

Table with n=5 discs.

	A	B	C
Initial Position	1,2,3,4,5	*	*
Move 1	2,3,4,5	*	1
Move 2	3,4,5	2	1
Move 3	3,4,5	1,2	*
Move 4	4,5	1,2	3
Move 5	1,4,5	2	3
Move 6	1,4,5	*	2,3
Move 7	4,5	*	1,2,3
Move 8	5	4	1,2,3

Move 9	5	1,4	2,3
Move 10	2,5	1,4	3
Move 11	1,2,5	4	3
Move 12	1,2,5	3,4	*
Move 13	2,5	3,4	1
Move 14	5	2,3,4	1
Move 15	5	1,2,3,4	*
Move 16	*	1,2,3,4	5
Move 17	1	2,3,4	5
Move 18	1	3,4	2,5
Move 19	*	3,4	1,2,5
Move 20	3	4	1,2,5
Move 21	3	1,4	2,5
Move 22	2,3	1,4	5
Move 23	1,2,3	4	5
Move 24	1,2,3	*	4,5
Move 25	2,3	*	1,4,5
Move 26	3	2	1,4,5
Move 27	3	1,2	4,5
Move 28	*	1,2	3,4,5
Move 29	1	2	3,4,5
Move 30	1	*	2,3,4,5
Move 31	*	*	1,2,3,4,5

b) Give a recurrence relation for a_n, the number of moves required to transfer n discs from one peg to another.

$a_n = 2a_{n-1} + 1$

c) Find an explicit formula for a_n

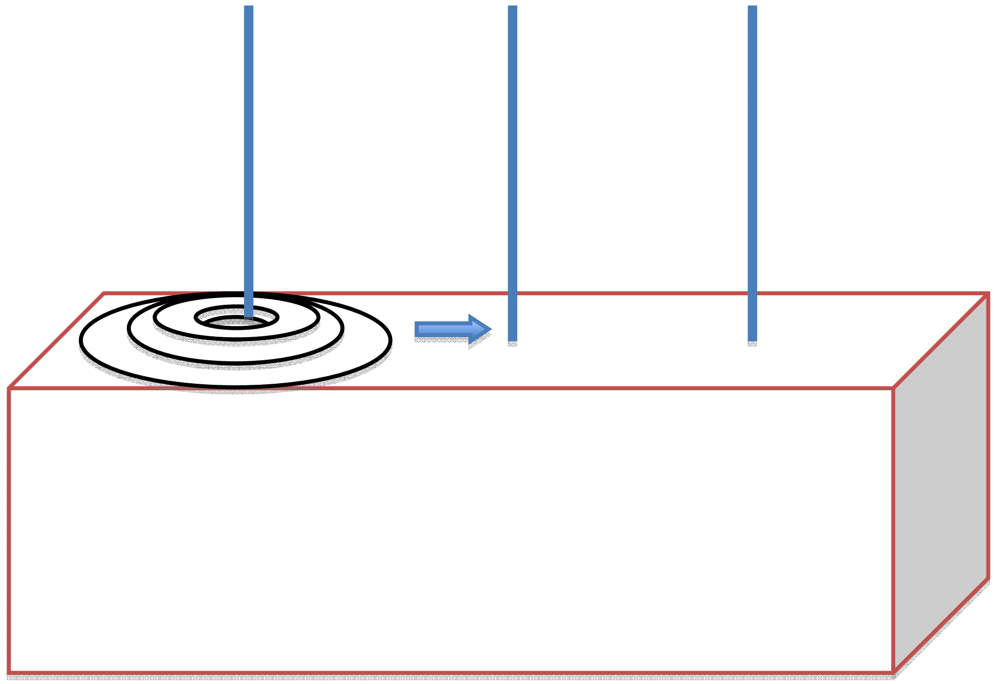
$a_n = 2^n - 1$ For all $n \geq 0$.

d) Suppose we can move a disc a second. Estimate the time required to transfer the discs if $n = 8$, $n = 16$, and $n = 64$ (Goodaire & Parmenter, 2006, p.147).

When $n = 8$ it takes $2^8 - 1 = 255$ seconds, which is ≈ 4 minutes.

When $n = 16$ it takes $2^{16} - 1 = 65535$ seconds, which is ≈ 18 hours.

When $n = 64$ it takes $2^{64} - 1 \approx 5.8 \times 10^{11}$ or 580,000,000,000 years.



5.6.4: Lesson Reflection

Lesson 5: Tower of Hanoi

Lesson 5 was definitely the most fun and accessible to the students. I made a Tower of Hanoi puzzle for each table group. It consisted of bricks of foam, pencils, and different size O washers from the hardware store. With minimal explanation each group set out to solve the puzzle. We talked about if you had $n=0$ discs then it would take 0 moves. Also, if you had $n=1$ discs then it would take 1 move to get to the next peg. With $n=3$ discs the students figured out how many moves fairly quickly. With $n=4$ and $n=5$ discs it took some more time, especially for the $n=5$ discs. In the end, all of the groups did $n=4$ discs and most of the groups finished $n=5$ discs. One student figured out the closed-form formula after the round of 4 discs. This activity was a fun and interesting closure to the unit on recurrence relations. It was also the last day of semester 1, so it was a perfect activity to do on the last day of class.

5.7: Back Substitution

5.7.1: Lesson Plan

Lesson 6: Back Substitution

Grade level: High School (9-12)	Subject: Advanced Algebra II	Unit Title: Techniques for creating explicit formulas from recurrence relations	Unit topic: Discrete Math-Recurrence Relations
Lesson# Six	Lesson Title: Back Substitution	Materials: Handout Graphing calculator	Overview and purpose: Use back substitution with recurrence relations to find explicit formulas

Objectives: Students will use the technique of back substitution to create explicit formulas from recurrence relations.
Information: This is another technique used to create an explicit formula. The instructor may choose to do this any time during the unit. Back substitution is a good way to show students patterns in sequences. They can visibly see number patterns sometimes "pop" out of the page. This technique will also help students comprehend series or even power series summations. Later in the unit, generating functions may be a topic that is chosen to challenge students.
Verification: Students will check each other for understanding during the class period by having their peers present their findings in front of the classroom. They will teach each other. Teacher will also check for understanding by prompting and promoting discussion and with open-ended questioning. Either at the end of the lesson or the following class period the teacher will collect student work and grade and give appropriate feedback. The following class period the class should review the previous days lesson to ensure comprehension and understanding before moving on to the next lesson.

Activity: With the handout students will do the problems using the back substitution technique.
Summary: Like stated earlier, back substitution is good for actually seeing patterns within the recurrence relation itself as well as the number sequence. It will also show students' "naturally" how geometric sums will be produced many times by using back substitution.

5.7.2: Student Handout

Name______________________________
Period__________

Lesson 6: Back Substitution

So far you have learned some techniques for taking a recurrence relation and converting it to an explicit formula. In lesson 2 we learned and practiced the characteristic polynomial. In lesson 3 we learned and practiced guess and check with induction to prove that our explicit formula works. Here in lesson 6 we will use another technique called back substitution. Back substitution is a good technique and starting point to learn about geometric sums. Below is an example on how back substitution works to help you solve some on your own.

Example

Given the recurrence relation $a_n = 2a_{n-1} + 1, n \geq 1$ with initial condition of $a_1 = 1$ find the explicit formula.

Solution:

$$a_n = 2a_{n-1} + 1$$
$$= 2(2a_{n-2} + 1) + 1$$
$$= 2(2(2a_{n-3} + 1)) + 2 + 1$$
$$= 2(2(2(2a_{n-4} + 1))) + 2^2 + 2 + 1$$
$$= 2(2(2(2(2a_{n-5} + 1)))) + 2^3 + 2^2 + 2 + 1$$
$$= 2^{n-1}a_1 + 2^{n-2} + 2^{n-3} + 2^{n-4} + 2^{n-5} \ldots + 2 + 1$$
$$= 2^{n-1}a_1 + \frac{2^{n-1} - 1}{2 - 1} \rightarrow 2^{n-1}(1) + 2^{n-1} - 1$$
$$\rightarrow (2^{n-1} + 2^{n-1} - 1) \rightarrow 2(2^{n-1} - 1) \rightarrow \therefore a_n = 2^n - 1, \forall n \geq 1$$

Exercises: Use the back substitution technique to convert each recurrence relation to an explicit formula. Make sure and test the initial conditions and a few more to see if your formula works.

1) Given the recurrence relation $5a_{n-1} - 1, n \geq 1$ with the initial condition of $a_1 = 1$, find the explicit formula.

2) Given the recurrence relation $3a_{n-1}+1, n \geq 1$ with the initial conditions of $a_1 = 1$, find the explicit formula.

3) Given the recurrence relation $a_n = 4a_{n-1}+1$ with the initial condition of $a_1 = 1$, find the explicit formula.

5.7.3: Instructor Solutions

Name______________________________
Period__________

Lesson 6: Back Substitution

So far you have learned some techniques for taking a recurrence relation and converting it to an explicit formula. In lesson 2 we learned and practiced the characteristic polynomial. In lesson 3 we learned and practiced guess and check with induction to prove that our explicit formula works. Here in lesson 6 we will use another technique called back substitution. Back substitution is a good technique and starting point to learn about geometric sums. Below is an example on how back substitution works to help you solve some on your own.

Example

Given the recurrence relation $a_n = 2a_{n-1} + 1, n \geq 1$ with initial condition of $a_1 = 1$ find the explicit formula.

Solution:

$$a_n = 2a_{n-1} + 1$$
$$= 2(2a_{n-2} + 1) + 1$$
$$= 2(2(2a_{n-3} + 1)) + 2 + 1$$
$$= 2(2(2(2a_{n-4} + 1))) + 2^2 + 2 + 1$$
$$= 2(2(2(2(2a_{n-5} + 1)))) + 2^3 + 2^2 + 2 + 1$$
$$= 2^{n-1}a_1 + 2^{n-2} + 2^{n-3} + 2^{n-4} + 2^{n-5} \ldots + 2 + 1$$
$$= 2^{n-1}a_1 + \frac{2^{n-1} - 1}{2 - 1} \rightarrow 2^{n-1}(1) + 2^{n-1} - 1$$
$$\rightarrow (2^{n-1} + 2^{n-1} - 1) \rightarrow 2(2^{n-1} - 1) \rightarrow \therefore a_n = 2^n - 1, \forall n \geq 1$$

Exercises: Use the back substitution technique to convert each recurrence relation to an explicit formula. Make sure and test the initial conditions and a few more to see if your formula works.

1) Given the recurrence relation $5a_{n-1} - 1, n \geq 1$ with the initial condition of $a_1 = \frac{3}{2}$, find the explicit formula.

Solution:

$$
\begin{aligned}
a_n &= 5a_{n-1} - 1 \\
&= 5(5a_{n-2} - 1) - 1 \\
&= 5(5(5a_{n-3} - 1)) - 5 - 1 \\
&= 5(5(5(5a_{n-4} - 1))) - 5^2 - 5 - 1 \\
&= 5\left(5(5(5(5a_{n-5} - 1)))\right) - 5^3 - 5^2 - 5 - 1 \\
&= 5^{n-1}a_1 - 5^{n-2} - 5^{n-3} - 5^{n-4} - 5^{n-5} \ldots - 5 - 1 \\
&= 5^{n-1}\left(\frac{3}{2}\right) - \left(5^{n-2} + 5^{n-3} + 5^{n-4} + 5^{n-5} \ldots + 5 + 1\right) \\
&= 5^{n-1}a_1 - \frac{5^{n-1} - 1}{5 - 1} \to 5^{n-1}\left(\frac{3}{2}\right) - \left(\frac{5^{n-1} - 1}{4}\right)
\end{aligned}
$$

$$
\to \frac{6 \bullet 5^{n-1} - 5^{n-1} + 1}{4} \to \frac{5^{n-1}(6-1)+1}{4} \to \therefore a_n = \frac{5^n + 1}{4}, \forall n \geq 1
$$

2) Given the recurrence relation $3a_{n-1} + 1, n \geq 1$ with the initial conditions of $a_1 = 1$, find the explicit formula.

Solution:

$$
\begin{aligned}
a_n &= 3a_{n-1} + 1 \\
&= 3(3a_{n-2} + 1) + 1 \\
&= 3(3(3a_{n-3} + 1)) + 3 + 1 \\
&= 3(3(3(3a_{n-4} + 1))) + 3^2 + 3 + 1 \\
&= 3\left(3(3(3(3a_{n-5} + 1)))\right) + 3^3 + 3^2 + 3 + 1 \\
&= 3^{n-1}a_1 + 3^{n-2} + 3^{n-3} + 3^{n-4} + 3^{n-5} \ldots + 3 + 1 \\
&= 3^{n-1}a_1 + \frac{3^{n-1} - 1}{3 - 1} \to 3^{n-1}(1) + \frac{3^{n-1} - 1}{2}
\end{aligned}
$$

$$
\to \left(3^{n-1} + \frac{3^{n-1} - 1}{2}\right) \to \left(\frac{2(3^{n-1}) + 3^{n-1} - 1}{2}\right)
$$

$$
\to \frac{3^{n-1}(2(1)+1) - 1}{2} \to \frac{(3^{n-1} \bullet 3) - 1}{2} \to \therefore a_n = \frac{3^n - 1}{2}, \forall n \geq 1
$$

3) Given the recurrence relation $a_n = 4a_{n-1} + 1, n \geq 1$ with the initial condition of $a_1 = 1$, find the explicit formula.

Solution:

$a_n = 4a_{n-1} + 1$

$= 4(4a_{n-2} + 1) + 1$

$= 4(4(4a_{n-3} + 1)) + (4) + 1$

$= 4(4(4(4a_{n-4} + 1))) + (4)^2 + (4) + 1$

$= 4\left(4(4(4(4a_{n-5} + 1)))\right) + (4)^3 + (4)^2 + (4) + 1$

$= (4)^{n-1}a_1 + (4)^{n-2} + (4)^{n-3} + (4)^{n-4} + (4)^{n-5} \ldots + (4) + 1$

$= (4)^{n-1}a_1 + \dfrac{(4)^{n-1} - 1}{4 - 1} \rightarrow (4)^{n-1}(1) + \dfrac{(4)^{n-1} - 1}{3}$

$\rightarrow \left((4)^{n-1} + \dfrac{(4)^{n-1} - 1}{3}\right) \rightarrow \left(\dfrac{3((4)^{n-1}) + (4)^{n-1} - 1}{3}\right)$

$\rightarrow \dfrac{4^{n-1}(3(1) + 1) - 1}{3} \rightarrow \dfrac{((4)^{n-1} \bullet 4) - 1}{3} \rightarrow \therefore a_n = \dfrac{(4)^n - 1}{3}, \forall n \geq 1$

5.7.4: Lesson Reflection

Lesson 6: Back Substitution

Back substitution is a great technique for students' to see patterns within the recurrence relation. By doing this technique students' will notice repetition in the recurrence relations that will allow them to solve for the closed-form formula. Back substitution may be done in any part of the curriculum. I think it would be helpful and provide a deeper understanding of recurrence relations if this lesson was done after an introduction to recurrence relations. By doing this lesson at the beginning of the unit it might help students' see the "big picture" rather than just doing computational math for instance, when trying to solve recurrence relations using the characteristic polynomial or other techniques.

5.8: Flagpoles

5.8.1: Lesson Plan

Lesson 7: Flagpoles

Grade level:	Subject:	Unit Title:	Unit topic:
Grade level: High School (9-12)	Subject: Advanced Algebra II	Unit Title: Techniques for creating explicit formulas from recurrence relations	Unit topic: Discrete Math-Recurrence Relations
Lesson# Seven	Lesson Title: Flagpoles	Materials: Handout Graphing calculator	Overview and purpose: Use alternate Pell formula to generate the Pell Numbers

Objectives: Students will use the alternate closed-form Pell formula to generate the Pell Numbers
Information: Each group will create their own flagpoles. The materials they will use can either be construction paper or colored pencils, markers or crayons to create various flagpoles. The colors will be red, blue and white. The red and blue flags will each be one foot tall and the white flag will be two feet tall. Students' will try and figure out what triples combination will work for a zero, one, two……etc., foot flagpole. Summing up all of these combinations in the alternate Pell formula will produce the Pell Numbers.
Verification: Students will check each other for understanding during the class period by having their peers present their findings in front of the classroom. They will teach each other. Teacher will also check for understanding by prompting and promoting discussion and with open-ended questioning. Either at the end of the lesson or the following class period the teacher will collect student work and grade and give appropriate feedback. The following period the class should review the previous days lesson to ensure comprehension and understanding before moving on to the next lesson.
Activity:

With sketches that mimic various flagpoles, students' will try and create multiple combinations of order and color on each of their flagpoles to generate the Pell Numbers

Summary:

Like the Tower of Hanoi, this should be a fun hands-on activity for students'. This activity will help students' with factorials, creating combinations and permutations.

5.8.2: Student Handout

Name________________________________
Period_________

Lesson 7: Flagpoles

In one of the previous lessons you used the characteristic polynomial to find a closed-form formula for the Pell Sequence. There exists other closed-form formulas including this one $p_n = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \frac{(i+j+k)!}{i!\,j!\,k!}$. This formula may seem intimidating but it is actually quite easy. First, factorials are repeated multiplication starting with the number shown and multiplying backwards in successive order. For example, $0! = 1 \rightarrow 1! = 1 \rightarrow 2! = 2 \bullet 1 \rightarrow 3! = 3 \bullet 2 \bullet 1 \rightarrow etc.....$

Remember, the Pell numbers are $p_0 = 1, p_1 = 2, p_2 = 5, p_3 = 12, p_4 = 29, p_5 = 70, etc$. If we think of the index number n as the size of a flagpole in feet and the Pell Number as the different ways to arrange the flags on the flagpole. The flags will be identified in the following way $i = red, j = blue, k = white$. Red and blue flags are each one-foot tall and white flags are two feet tall. For example $p_0 = 1$ would be a zero foot flagpole with only one way to arrange those flags on the flagpole. For question 1 you will do computations like the one below. The computation will look like:

$$p_n = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \frac{(i+j+k)!}{i!\,j!\,k!} \rightarrow p_0 = 1$$

$$i + j + k = n \rightarrow 0 + 0 + 0 = 0 \rightarrow$$

$$(i,j,k) = (0,0,0) \rightarrow \frac{(0+0+0)!}{0!0!0!} = \frac{1}{1} = 1$$

Basically the only triple (i,j,k) that will work for a zero foot flagpole is $(0,0,0)$ which is only one combination. So when you try $p_1 = 2$, there will be a one-foot flagpole with two combinations that will work for this case. Then for $p_2 = 5$ there will be a two foot flagpole with 5 combinations that will work.

For question 2 you will create sketches of flags on flagpoles according to the results you got on question 1. For example we know that $p_3 = 12$, which means for a 3 foot flagpole (the index number) there are 12 possible combinations. If you use $R = red, B = blue, W = white$ then you will group them as such:

$$WB \quad WR \rightarrow 2$$
$$BBB \quad BW \quad BRR \quad BRB \quad BBR \rightarrow 5$$
$$RBB \quad RW \quad RRR \quad RRB \quad RBR \rightarrow 5$$

If you notice each flag combination is classified by the first letter in each combination which is the color of the flag at the top of the flagpole. There are 2 White, 5 Blue and 5 Red combinations. The sum of these 3 numbers is 12 which is indeed $p_3 = 12$. Another thing to notice is that the number 2 is actually $p_1 = 2$ and the number 5 which occurs twice is $p_2 = 5$. Therefore with this information if we use the Pell Sequence recurrence relation of $p_n = 2p_{n-1} + p_{n-2}$ then we have

$p_3 = 2p_2 + p_1 \rightarrow 2(5) + 2 = 12$

There are two parts to the exercise of which each will reinforce the other. With red, blue, and white construction paper or crayons, markers and colored pencils you will create flags and flagpoles that will represent each of the combinations for each Pell Number. You will also use the formula to calculate the Pell Numbers that should match the different combinations of flags and flagpoles you are making.

Exercises:

1) Using the alternate Pell formula $p_n = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \frac{(i+j+k)!}{i!\,j!\,k!}$ to calculate the first six Pell Numbers.

2) Now that you have calculated the first six Pell Numbers:

a) Create by sketching all of the 4-foot flagpoles. How many are there?

b) From your results in question a) group each flagpole according to the color at the top of each flagpole. How many are in each category.

c) What pattern do you notice? Can you write a recurrence relation from this pattern?

5.8.3: Instructor Solutions

Name________________________________
Period_________

Lesson 7: Flagpoles

In one of the previous lessons you used the characteristic polynomial to find a closed-form formula for the Pell Sequence. There exists other closed-form formulas including this one $p_n = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \frac{(i+j+k)!}{i!\,j!\,k!}$. This formula may seem intimidating but it is actually quite easy. First, factorials are repeated multiplication starting with the number shown and multiplying backwards in successive order. For example, $0! = 1 \rightarrow 1! = 1 \rightarrow 2! = 2 \bullet 1 \rightarrow 3! = 3 \bullet 2 \bullet 1 \rightarrow etc.....$

Remember, the Pell numbers are $p_0 = 1, p_1 = 2, p_2 = 5, p_3 = 12, p_4 = 29, p_5 = 70, etc$. If we think of the index number n as the size of a flagpole in feet and the Pell Number as the different ways to arrange the flags on the flagpole. The flags will be identified in the following way $i = red, j = blue, k = white$. Red and blue flags are each one-foot tall and white flags are two feet tall. For example $p_0 = 1$ would be a zero foot flagpole with only one way to arrange those flags on the flagpole. For question 1 you will do computations like the one below. The computations will look like:

$$p_n = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \frac{(i+j+k)!}{i!\,j!\,k!} \rightarrow p_0 = 1$$

$$i + j + k = n \rightarrow 0 + 0 + 0 = 0 \rightarrow$$

$$(i,j,k) = (0,0,0) \rightarrow \frac{(0+0+0)!}{0!0!0!} = \frac{1}{1} = 1$$

Basically the only triple (i,j,k) that will work for a zero foot flagpole is $(0,0,0)$ which is only one combination. So when you try $p_1 = 2$, there will be a one-foot flagpole with two combinations that will work for this case. Then for $p_2 = 5$ there will be a two foot flagpole with 5 combinations that will work.

For question 2 you will create sketches of flags on flagpoles according to the results you got on question 1. For example we know that $p_3 = 12$, which means for a 3 foot flagpole (the index number) there are 12 possible combinations. If you use $R = red, B = blue, W = white$ then you will group them as such:

$$WB \quad WR \rightarrow 2$$

$$BBB \quad BW \quad BRR \quad BRB \quad BBR \rightarrow 5$$

$$RBB \quad RW \quad RRR \quad RRB \quad RBR \rightarrow 5$$

If you notice each flag combination is classified by the first letter in each combination which is the color of the flag at the top of the flagpole. There are 2 White, 5 Blue and 5 Red combinations. The sum of these 3 numbers is 12 which is indeed $p_3 = 12$. Another thing to notice is that the number 2 is actually $p_1 = 2$ and the number 5 which occurs twice is $p_2 = 5$. Therefore with this information if we use the Pell Sequence recurrence relation of $p_n = 2p_{n-1} + p_{n-2}$ then we have $p_3 = 2p_2 + p_1 \rightarrow 2(5) + 2 = 12$

There are two parts to the exercise of which each will reinforce the other. With red, blue, and white construction paper or crayons, markers and colored pencils you will create flags and flagpoles that will represent each of the combinations for each Pell Number. You will also use the formula to calculate the Pell Numbers that should match the different combinations of flags and flagpoles you are making.

Instructor Solutions

Exercises:

1) Using the alternate Pell formula $p_n = \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \frac{(i+j+k)!}{i!\,j!\,k!}$ to calculate the first six Pell Numbers.

$$p_0 = 1$$
$$i + j + 2k = n$$
$$i + j + 2k = 0$$
$$(i, j, k)$$
$$(0,0,0)$$
$$\frac{(0+0+0)!}{0!0!0!} = \frac{1}{1} = 1$$

$$p_1 = 2$$
$$i + j + 2k = n$$
$$i + j + 2k = 1$$
$$(i, j, k)$$
$$(0,1,0) + (1,0,0)$$
$$\frac{(0+1+0)!}{0!1!0!} + \frac{(1+0+0)!}{1!0!0!}$$
$$\frac{1}{1} + \frac{1}{1} = 1 + 1 = 2$$

$$p_2 = 5$$
$$i + j + 2k = n$$
$$i + j + 2k = 2$$
$$(i, j, k)$$
$$(2,0,0) + (1,1,0) + (0,2,0) + (0,0,1)$$
$$\frac{(2+0+0)!}{2!0!0!} + \frac{(1+1+0)!}{1!1!0!} + \frac{(0+2+0)!}{0!2!0!} + \frac{(0+0+1)!}{0!0!1!}$$
$$\frac{2}{2} + \frac{2}{1} + \frac{2}{2} + \frac{1}{1} = 1 + 2 + 1 + 1 = 5$$

$p_3 = 12$

$i + j + 2k = n$

$i + j + 2k = 3$

(i, j, k)

$(0,1,1) + (1,0,1) + (2,1,0) + (1,2,0) + (3,0,0) + (0,3,0)$

$$\frac{(0+1+1)!}{0!1!1!} + \frac{(1+0+1)!}{1!0!1!} + \frac{(2+1+0)!}{2!1!0!} + \frac{(1+2+0)!}{1!2!0!} + \frac{(3+0+0)!}{3!0!0!} + \frac{(0+3+0)!}{0!3!0!}$$

$2 + 2 + 3 + 3 + 1 + 1 = 12$

$p_4 = 29$

$i + j + 2k = n$

$i + j + 2k = 4$

(i, j, k)

$(0,2,1) + (2,0,1) + (1,1,1) + (0,0,2) + (1,3,0) + (3,1,0) + (4,0,0) + (0,4,0) + (2,2,0)$

$$\frac{(0+2+1)!}{0!2!1!} + \frac{(2+0+1)!}{2!0!1!} + \frac{(1+1+1)!}{1!1!1!} + \frac{(0+0+2)!}{0!0!2!} + \frac{(1+3+0)!}{1!3!0!} + \frac{(3+1+0)!}{3!1!0!} \frac{(4+0+0)!}{4!0!0!} + \frac{(0+4+0)!}{0!4!0!}$$
$$+ \frac{(2+2+0)!}{2!2!0!}$$

$3 + 3 + 6 + 1 + 4 + 4 + 1 + 1 + 6 = 29$

$p_5 = 70$

$i + j + 2k = n$

$i + j + 2k = 5$

(i, j, k)

$(0,1,2) + (1,0,2) + (4,1,0) + (1,4,0) + (3,2,0) + (2,3,0) + (3,0,1) + (0,3,1) + (5,0,0)$

$(0,5,0) + (2,1,1) + (1,2,1)$

$$\frac{(0+1+2)!}{0!1!2!} + \frac{(1+0+2)!}{1!0!2!} + \frac{(4+1+0)!}{4!1!0!} + \frac{(1+4+0)!}{1!4!0!} + \frac{(3+2+0)!}{3!2!0!} + \frac{(2+3+0)!}{2!3!0!} \frac{(3+0+1)!}{3!0!1!} + \frac{(0+3+1)!}{0!3!1!}$$
$$+ \frac{(5+0+0)!}{5!0!0!} + \frac{(0+5+0)!}{0!5!0!} + \frac{(2+1+1)!}{2!1!1!} + \frac{(1+2+1)!}{1!2!1!}$$

$3 + 3 + 5 + 5 + 10 + 10 + 4 + 4 + 1 + 1 + 12 + 12 = 70$

2) Now that you have calculated the first six Pell Numbers:

a) Create by sketching all of the 4-foot flagpoles. How many are there?
There are 29 possibilities which is $p_4 = 29$. Below are the possible combinations.

WW WRB WBR WBB WRR
BBW BRW BBRR BBRB BBBR BBBB BRBR BRBB BRRR BRRB BWR BWB
RRW RBW RRBB RRBR RRRB RRRR RBRB RBRR RBBB RBBR RWB RWR

b) From your results in question a) group each flagpole according to the color at the top of each flagpole. How many are in each category.

WW WRB WBR WBB WRR
BBW BRW BBRR BBRB BBBR BBBB BRBR BRBB BRRR BRRB BWR BWB
RRW RBW RRBB RRBR RRRB RRRR RBRB RBRR RBBB RBBR RWB RWR

c) What pattern do you notice? Can you write a recurrence relation from this pattern?

There are 5 combinations with white on top, 12 with blue on top and 12 with red on top. Notice that there are two $p_3 = 12$ and one $p_2 = 5$. So $p_4 = 2p_3 + p_2 \rightarrow 2(12) + 5 = 29$ which is the desired result.

Notes:

Use the (i, j, k) triples from exercise 1 to construct the various flags and flagpoles. Remember red and blue flags are one foot tall and white flags are two feet tall. The corresponding colors of the flags are as follows: $i = red, j = blue, k = white$.

One thing to notice closely is for triples that might represent more than one way to arrange the colored flags on the flagpole. For example in $p_2 = 5$, there are only 4 triples shown to calculate to get to the number 5. However, the triple $(1,1,0)$ has two ways to arrange the flags on the flagpole. One way is to put a blue flag on top and red on bottom and the other way is to put a red flag on top and a blue flag on the bottom.

5.8.4: Lesson Reflection

<u>Lesson 7: Flagpoles</u>

The flagpoles lesson is similar to the Tower of Hanoi in that students' are able to use hands on manipulatives to make more sense of recurrence relations. This lesson will teach and help student's practice factorials, permutations and combinations. The actual alternate Pell Formula will help students' calculate the actual terms of the Pell Sequence. The creative part is coming up with the combinations that will work to generate the desired index number. Students' will be able to visually test out there (i, j, k) combinations to see how many different ways they can place red, blue and white flags on the flagpole. This lesson might be done before or after the Pell Sequence lesson.

5.9 Summary of Curriculum

Overall the curriculum component was a success and a learning experience for my students and myself. I think it went well overall because since we have known each other for so long there was a level of trust to go down this path that was unfamiliar and challenging to them. I was fortunate because most of the students in this class are very bright, positive, and motivated. Doing the chapter 1 unit from our Algebra 2 textbook was a good thing because it gave them some confidence and familiarity with recurrence relations. The introduction lesson 1 went well because it was based off of the chapter we had just covered in class the previous 2-3 weeks. The Characteristic Polynomial was fairly successful because there was familiarity and knowledge about quadratics. Lesson 3 part 1 was also fairly successful after some initial modeling of factoring of terms with unfamiliar notation. Lesson 3 part 2 was probably the least successful. None of the students had ever seen induction and just had a hard time grasping the topic conceptually. It also did not help that two-thirds of the class was missing all of that period because of a school activity. The Pell Sequence was successful also because they used the characteristic polynomial again as well as the closed form factorial formula which they found interesting and amazing that the formula could get each number of the Pell Sequence. The tower of Hanoi was probably the most fun and light.

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John Pell (1611-1685)

Lesson 1: Writing Recursive Definitions

Give definitions for the following words below as well as an example.

1) Recursion: process in which each step of a pattern is dependant on the step or steps that come before it.

2) Sequence: list of numbered, in order.

3) Term: Each number in a sequence.

4) general term: u_n

5) recurrence relation (recursive formula): formula that defines a sequence

6) initial condition(s): starting value

8) What is an arithmetic sequence? Give the symbolic form and what each part means.

Each term is equal to the previous term plus a constant. $u_0, = 1$ $u_1 = 1$

9) What is a geometric sequence? Give the symbolic form and what each part means.

Each term is equal to the previous term, multiplied by a constant. $u_n = r \cdot u_{n-1}$

10) What is the symbolic form of a shifted geometric sequence? What is the long-run value?

$$u_n = r \cdot u_{n-1} + d$$

[redacted]mula to generate each [redacted] for the first term. Then find the next 3 terms in the sequence.

a) 3, 7, 11, 15, …… 19, 23, 27

+4 $\boxed{u_n = u_{n-1} + 4}$

b) 15, 5, −5, −15, …….. −25, −35, −45

−10 $\boxed{u_n = u_{n-1} - 10}$

c) .3, .03, .003, .0003, ……… .00003, .000003, .0000003

$\frac{1}{10}$ $\boxed{u_n = \frac{1}{10} \cdot u_{n-1}}$

d) 100, 150, 225, 337.5, ……

$\frac{3}{2}$ $\boxed{u_n = \frac{3}{2} \cdot u_{n-1}}$

12) List the first five terms of the sequence

a) $u_1 = -4$

$u_n = u_{n-1} - 1.5$ where $n \geq 2$

$u_2 = u_1 - 1.5$

$-4 - 1.5 = -5.5 = u_2$

−4, −5.5, −7, −8.5, −10

b) $u_1 = 1$

$u_n = 3u_{n-1} - 2$ where $n \geq 2$

$u_3 = 3u_2 - 2$

$3 \cdot 1 - 2$

$u_2 = 3u_1 - 2$

$3 \cdot 1 - 2$

$u_2 = 1,\ u_3 = 1,\ u_4 = 1,\ u_5 = 1,\ u_6 = 1$

c) $u_0 = 256$

$u_n = 0.75u_{n-1}$

$u_1 = 0.75\,u_0$

$.75 \cdot 256$

$u_1 = 192$

$u_2 = 0.75\,u_1$

$.75 \cdot 192$

$u_1 = 192,\ u_2 = 144,\ u_3 = 108$

$u_4 = 81,\ u_5 = 60.75$

13) Application: A nursery owns 7000 Japanese maple trees. Each year the nursery plans to sell 12% of it's trees and plant 600 new ones.

a) Write a recursive definition that represents the nursery's tree stock.

$\cancel{u_n = u_{n-1} + .88}$ $u_{n+1} \cdot .88 + 600 = u_n$

b) Find the number of pine trees owned by the nursery after 10 years.

c) At what point will the amount of trees planted equal the amount of trees sold by the nursery?

14) Here are some recurrence relations that are neither arithmetic nor geometric. List the first 6 terms of each sequence. Instead of u_n use a_n.

a) $a_1 = \frac{3}{2}$

$a_n = 5a_{n-1} - 1$ for $n \geq 2$

$a_1 = \frac{3}{2}, a_2 = \frac{13}{2}, a_3 = \frac{63}{2}, a_4 = \frac{311}{2}$

$a_2 = 5a_1 - 1 = \frac{13}{2}, \quad a_3 = 5a_2 - 1 = \frac{63}{2}, \quad a_4 = 5a_3 - 1$

$\frac{63}{2}$

b) $a_0 = -3,\ a_1 = -2$

$a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$

$a_2 = 5a$

?

c) $a_1 = 10,\ a_2 = 29$

$a_{n+1} = 7a_n - 10a_{n-1}$ for $n \geq 2$

$7 \cdot 29 - 10 \cdot 10 = 29$

$10, 29, 103, 431, 1987$

15) Write a recurrence relation for the following sequences. Use a_1 for the first term in the sequence

a) 1,1,2,3,5,8,13,.. 21, 34, 55, 89

$u_n = u_n \cdot u_{n-1}$

$u_3 = u_2 + u_1$

$u_3 = 1 + 1 = 2$

b) 1,4,9,16, ….. 25, 36, 49, 64, 81

$a_n = n^2$

c) 1,2,6,~~10~~ 24, ……120, 720

$A_n = n!$

d) 4,1,3, -2,5, -7,12, -19,31, ……

$a_{n-2} - a_{n-1} = a_n$

$a_1 = 4$

$n \geq 3$

$a_2 = 1$

Lesson 2: Characteristic Polynomial: A technique for converting recurrence relations to a closed form formula.

The characteristic polynomial is a technique used for solving recursively-defined sequences. Recurrence relations are useful when trying to find patterns in number sequences. Remember though, a drawback when using recurrence relations has been given in the example of, "What if I want to find the 100th term in the sequence?" In order to do that you need to know the 99th term which means you have to do recursion 99 times.

The characteristic polynomial is the first technique you will learn to find what is called a "closed form formula" or an "explicit formula" for a recurrence relation. Creating this will allow you to easily find the 100th term of the sequence as well as any other term in the sequence.

Example: Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = -3$, $a_1 = -2$ an $n \geq 2$

You can think of the recurrence relation in terms of a quadratic expression of the form $ax^2 + bx + c$ where a, b and c are constants (numbers). In other words a_n is the ax^2 term, $5a_{n-1}$ is the bx term and $6a_{n-1}$ is the c term.

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$-5a_{n-1} + 6a_{n-2}$ $\quad$ $-5a_{n-1} + 6a_{n-2}$ (Subtract from both sides)

$a_n - 5a_{n-1} + 6a_{n-2} = 0$ (Which converts to) $x^2 - 5x + 6$

$(x-2)(x-3)$ (Which gives us roots of) $x_1 = 2$ and $x_2 = 3$

(Next we put these roots into the equation) $a_n = c_1(2^n) + c_2(3^n)$

With the first initial condition of $a_0 = -3$ we can substitute 0 for n

$a_n = c_1(2^n) + c_2(3^n)$ (Becomes) $a_0 = c_1(2^0) + c_2(3^0)$ (Which then becomes). $-3 = c_1 + c_2$ (Remember anything to the zero power is one.)

The second initial condition is $a_1 = -2$. We can substitute 1 for n

$a_n = c_1(2^n) + c_2(3^n)$ (Becomes) $a_1 = c_1(2^1) + c_2(3^1)$ (Which then becomes) $-2 = 2c_1 + 3c_2$

(We now have two equations with two variables)

$-3 = c_1 + c_2$
$-2 = 2c_1 + 3c_2$

(Solve using elimination or substitution we get)

$c_1 = -7$ And $c_2 = 4$

(Substituting) $c_1 = -7$ and $c_2 = 4$ (into) $a_n = c_1(2^n) + c_2(3^n)$ (yields)

$a_n = -7(2^n) + 4(3^n)$ (Which is the closed form formula or our solution)

If you test the formula you will see $a_0 = -3$, $a_1 = -2$ and the 100th term is 206151008300000000000000000000000000000000000000000!

Exercises: For each recurrence relation find the closed form formula. After you create your formula make sure and test the initial conditions to see if it works.

1) $a_n = 4a_{n-1}$ given $a_0 = 1$, $a_1 = 4$ where $n \geq 2$

$a_n = 4a_{n-1}$ $\qquad a_0 = 1$, $a_1 = 4$

$(x^2 - 4x)$

$1 = c_1 + c_2$

$4 = 4c_1 + c_2$

$x(x-4) = 0$

$(4, 0)$

$\boxed{a_n = 4^n}$ ✓

$a_n = c_1(4^n) + c_2(0^n)$

$a_0 = c_1 + c_2, \quad a_1 = c_1(4^1) + c_2(0^1)$

$\boxed{1 = c_1 + c_2}$ $\quad$ $\boxed{4 = 4c_1 + c_2}$

2) $a_n = -2a_{n-1} + 15a_{n-2}$ given $a_0 = 1$, $a_1 = 1$ where $n \ge 2$

$a_0 = 1 \quad a_1 = 1$

$x^2 + 2x - 15 = 0$

$(x+5)(x-3)$

$x^2 - 3x + 5x - 15$

$x + 2x - 15$

$a_n = C_1(-5^n) + C_2(3^n)$

$1 = C_1 + .75$

$-.75 \quad -.75$

$\boxed{1 = C_1 + C_2}$

$\boxed{1 = -5C_1 + 3C_2}$

$.5\ 1 = C_1 + C_2 \qquad 5 = 5C_1 + 5C_2$

$1 = -5C_1 + 3C_2 \qquad 1 = -5C_1 + 3C_2$

$C_2 = .75 \qquad \frac{6}{8} = \frac{8C_2}{8}$

$C_1 = .25$

$\boxed{a_n = .25(-5^n) + .75(3^n)}$ ✓

3) $a_n = -6a_{n-1} + 7a_{n-2}$ given $a_0 = 32$, $a_1 = -17$

$x^2 + 6x - 7$

$(x+7)(x-1)$

$x^2 - 1x + 7x - 7$

$x^2 + 6x - 7$

$32 = 6.125 + C_2$

$-6.125 \quad -6.125$

$-1\ 32 = C_1 + C_2$

$-17 = -7C_1 + C_2$

$a_n = C_1(-7^n)\ C_2(1^n)$

$a_0 = C_1(-7^0)\ C_2(1^0)$

$-32 = -1C_1$

$+ -17 = -7C_1 +$

$\boxed{32 = C_1 + C_2}$

$\frac{-49}{-8} = \frac{-8C_1}{-8} \quad C_1 = 6.125$

$a_1 = C_1(-7^1)\ C_2(1^1)$

$C_2 = 25.875$ ✓

$\boxed{-17 = -7C_1 + C_2}$

$\boxed{a_n = 6.125(-7^n) + 25.875(1^n)}$

(4)

$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a=1$
$b=8$
$c=1$

4) $a_n = -8a_{n-1} - a_{n-2}$ given $a_0 = 0$ and $a_1 = 1$

$X^2 + 8x + 1$ $\quad X = \frac{-8 \pm \sqrt{8^2 - 4ac}}{2}$

$X = 4$

$X = -4 + \sqrt{15}$

$X = -4 - \sqrt{15}$

$an = C_1(-4+\sqrt{15}) + C_2(-4-\sqrt{15})$

If you Keep going
You will get
$X = -4 \pm \sqrt{15}$
then go fromthere

5) $a_{n+1} = 7a_n - 10a_{n-1}$ given $a_0 = 10$ and $a_1 = 29$

$X^2 - 7 + 10$ $\quad x^2 - 2x - 5x + 10$

$(X-5)(X-2)$

$an = C_1(+5^n) + C_2(+2^n)$

$a_0 = C_1(+5^0) + C_2(+2^0)$

$\boxed{10 = C_1 + C_2}$

$a_1 = C_1(+5^1) + C_2(+2^1)$

$\boxed{29 = +5C_1 + 2C_2}$

$10 = 3 + C_2$
$-3 \quad -3$

$\cdot 2 \quad 10 = C_1 + C_2$
$29 = 5C_1 + 2C_2$

$-20 = -2C_1 - \cancel{2C_2}$
$29 = 5C_1 \ \cancel{2C_2}$

$\frac{9}{3} = \frac{3C_1}{3}$ $\quad C_1 = 3$

$C_2 = 7$

$an = 3(5^n) + 7(2^n)$ ✓

Lesson 3: Checking the Explicit Formula and Guess and Check with Induction:

Part 1: Guess and Check: In lesson 2 we used the characteristic polynomial to find an explicit formula for a recurrence relation. We will now check the explicit formula with the recurrence relation to see if it will work for any of the natural numbers from $[1,\infty)$

Example 1(checking the explicit formula): We have already found from example 1 lesson 2 that the explicit formula for $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = -3$, $a_1 = -2$, $n \geq 2$ is $a_n = -7(2^n) + 4(3^n)$
This first example is a check to see if the closed form formula will actually work as defined from the recurrence relation.

First since $a_n = -7(2^n) + 4(3^n)$ we will substitute this expression into

$a_n = 5a_{n-1} - 6a_{n-2}$ giving us $-7(2^n) + 4(3^n) = 5a_{n-1} + 6a_{n-2}$

Next if $a_n = -7(2^n) + 4(3^n)$ Then $a_{n-1} = -7(2^{n-1}) + 4(3^{n-1})$ and $a_{n-2} = -7(2^{n-2}) + 4(3^{n-2})$

We substitute to get the following massive equation of:
$-7(2^n) + 4(3^n) = 5[-7(2^{n-1}) + 4(3^{n-1})] - 6[-7(2^{n-2}) + 4(3^{n-2})]$

(note the first expression in brackets is a_{n-1} and the second is a_{n-2})

Next we can get the "massive equation" all on the left side.
$-7(2^n) + 4(3^n) - 5[-7(2^{n-1}) + 4(3^{n-1})] + 6[-7(2^{n-2}) + 4(3^{n-2})] = 0$

Then we factor out
$2^{n-2} * 3^{n-2}\{-7(2^2) + 4(3^2) - 5[-7(2^1) + 4(3^1)] + 6[-7(2^0) + 4(3^0)]\ \} = 0$

$2^{n-2} * 3^{n-2}\{0\ \} = 0$ (Note: all of the numbers in the braces above go to zero!)
$0 = 0$ Check! This identifies that your closed form formula is valid!

[redacted] each closed form formula by substituting it into the recurrence relation [redacted]

1) Recurrence relation $a_n = 4a_{n-1}$ given $a_0 = 1$, $a_1 = 4$ where $n \ge 2$

closed form formula $a_n = 4^n$

$4^n = 4[4^{n-1}]$

$-4(4^{n-1})$

$4^n - 4[4^{n-1}] = 0$

$4^{n-1}(4^1 - 4) = 0$

$4^{n-1}(0) = 0$

2) $a_n = -2a_{n-1} + 15a_{n-2}$ given $a_0 = 1$, $a_1 = 1$ where $n \ge 2$

$a_n = \frac{1}{2}(-5^n) + \frac{1}{2}(3^n)$

$\cancel{-2a_{n-1} + 15a_{n-2}}$

$\frac{1}{2}(-5^n) + \frac{1}{2}(3^n) = -2[\frac{1}{2}(-5^{n-1}) + \frac{1}{2}(3^{n-1})] + 15[\frac{1}{2}(-5^{n-2}) + \frac{1}{2}(3^{n-2})]$

$\frac{1}{2}(-5^n) + \frac{1}{2}(3^n) + 2[\frac{1}{2}(-5^{n-1}) + \frac{1}{2}(3^{n-1})] - 15[\frac{1}{2}(-5^{n-2}) + \frac{1}{2}(3^{n-2})] = 0$

$-5^{n-2} \cdot 3^{n-2}(\frac{1}{2}(-5^2) + \frac{1}{2}(3^2) + 2[\frac{1}{2}(-5^1) + \frac{1}{2}(3^1)] - 15[\frac{1}{2}(-5^0) + \frac{1}{2}(3^0)] = 0$

$-5^{n-2} \cdot 3^{n-2}(12.5 + 4.5) + 2[-2.5 + 1.5] - 15[\frac{1}{2} + \frac{1}{2}] = 0$

$-5^{n-2} \cdot 3^{n-2}[17 - 2 - 15] = 0$

$-5^{n-2} \cdot 3^{n-2}[0] = 0$

3) $a_n = -6a_{n-1} + 7a_{n-2}$ given $a_0 = 32$, $a_1 = -17$

$$a_n = \frac{49}{8}(-7^n) + \frac{207}{8}(1^n)$$

$$\frac{49}{8}(-7^n) + \frac{207}{8}(1^n) + 6\left[\frac{49}{8}(-7^{n-1}) + \frac{207}{8}(1^{n-1})\right] - 7\left[\frac{49}{8}(-7^{n-2}) + \frac{207}{8}(1^{n-2})\right] = 0$$

$$-7^{n-2} \cdot 1^{n-2} \cdot \frac{49}{8}(-7^2) + \frac{207}{8}(1^2) + 6\left[\frac{49}{8}(-7^1) + \frac{207}{8}(1^1)\right] - 7\left[\frac{49}{8}(-7^0) + \frac{207}{8}(1^0)\right] = 0$$

$$-7^{n-2} \cdot 1^{n-2} \cdot \frac{49}{8} \cdot 49 + \frac{207}{8} + 6\left[-42.875 + \frac{207}{8}\right] - 7\left[\frac{49}{8} + \frac{207}{8}\right] = 0$$

$$-7^{n-2} \cdot 1^{n-2} \quad 300.125 + \frac{207}{8} - 257.25 + 155.25 - 224 = 0$$

$$-7^{n-2} \cdot 1^{n-2} \quad 224 - 224 = 0$$

$$-7^{n-2} \cdot 1^{n-2} \quad 0 = 0$$

4) $a_n = -8a_{n-1} - a_{n-2}$ given $a_0 = 0$ and $a_1 = 1$

$$\frac{1}{2\sqrt{15}}(-4+\sqrt{15})^n - \frac{1}{2\sqrt{15}}(-4-\sqrt{15})^n$$

5) $a_{n+1} = 7a_n - 10a_{n-1}$ given $a_0 = 10$ and $a_1 = 29$

$a_n = \frac{3}{5}(5^n) + \frac{21}{6}(2^n)$

$$\frac{3}{5}(5^n) + \frac{21}{6}(2^n) = 7\left[\frac{3}{5}(5^{n-1}) + \frac{21}{6}(2^{n-1})\right] - 10\left[\frac{3}{5}(5^{n-2}) + \frac{21}{6}(2^{n-2})\right]$$

$$\frac{3}{5}(5^n) + \frac{21}{6}(2^n) - 7\left[\frac{3}{5}(5^{n-1}) + \frac{21}{6}(2^{n-1})\right] + 10\left[\frac{3}{5}(5^{n-2}) + \frac{21}{6}(2^{n-2})\right] = 0$$

$$5^{n-2} \cdot 2^{n-2}\ \frac{3}{5}(5^2) + \frac{21}{6}(2^2) - 7\left[\frac{3}{5}(5^1) + \frac{21}{6}(2^1)\right] + 10\left[\frac{3}{5}(5^0) + \frac{21}{6}(2^0)\right] = 0$$

$$5^{n-2} \cdot 2^{n-2}\ 15 + 14 - 7[3 + 7] + 10\left[\frac{3}{5} + \frac{21}{6}\right] = 0$$

$$5^{n-2} \cdot 2^{n-2}\ 29 - 70 + 41 = 0$$

$$5^{n-2} \cdot 2^{n-2}\ 0 = 0$$

Part 2: Guess and Check with Induction - Induction is a technique to prove that recurrence relations that are converted to a closed form formula will work for any of the natural numbers $[1,\infty)$.

In lesson 2 you learned a technique for converting recurrence relations to closed form formulas known as the characteristic polynomial. From the example in lesson 2 your were shown how to solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = -3$, $a_1 = -2$ an $n \geq 2$ and get the closed form formula of $a_n = -7(2^n) + 4(3^n)$.

After that you tested the initial conditions and maybe several other terms to see if the closed form formula works. The problem is, how do you know that your formula will work for all of the natural numbers? The Principle of Mathematical Induction proves that your formula will work for all the natural numbers.

(The natural numbers (also known as the counting numbers) go from $[1,\infty)$)

The Principle of Mathematical Induction states:

If P_n is true, and

a) P_1 is true, and,

b) assuming P_k is true implies that P_{k+1} is true,

then P_n must be true for all positive integers n

Example 2: Induction

Let $a_1, a_2, a_3,$ .... be the sequence defined by $a_1 = 1$, $a_{k+1} = 3a_k$ for $k \geq 1$. Prove that $a_n = 3^{n-1}$ for all $n \geq 1$

Proof:

1) base case for $n = 1$ $a_1 = 3^{1-1} = 3^0 = 1$ check

2) Assume $a_n = 3^{n-1}$, then $a_{n+1} = 3^{(n+1)-1} = 3^n$ then $a_n = 3^n$ is true

   Consider $a_{k+1} = 3a_k$ which is given, then $a_{n+1} = 3a_n = 3^1(3^{n-1}) = 3^{1+n-1} = 3^n$ check

   Therefore, by induction $a_n = 3^{n-1}$ is true for all $n \geq 1$

Exercises: For excercises #1-3 use induction to prove the explicit formula is correct. For problems 4 and 5 find the first 6 terms of the sequence, guess a formula for the recurrence relation and use induction to prove your formula is correct.

1) Prove that the sum of n consecutive positive odd integers is n^2. In other words prove that $1+3+5+\ldots\ldots+(2n-1)=n^2$

Base Case: $n=1$

$2\cdot1-1=1^2$

$2-1=1^2$

$1=1$

W.t.S $(2k-1)+2(k+1)-1=k^2+2(k+1)-1$

$(2k-1)+2(k+1)-1=k^2+2(k+1)-1$

$=k^2+2k+2-1$

$=k^2+2k+1$

$=(k+1)(k+1)$

$=(k+1)^2$

$a_n=\frac{n(n+1)}{2}$ $a_1=1$ $a_2=3$ $a_3=6$

2) Prove that $1+2+4+8+\ldots\ldots+n=\frac{n(n+1)}{2}$

$a_{n+1}=\frac{(n+1)(n+2)}{2}$

1) Base case when $n=1$

$1=\frac{1(1+1)}{2}$ $1=\frac{2}{2}$ $1=1$

$n=k$

$k=\frac{k(k+1)}{2}$

Add $k+1$ to both sides

$k+k+1=\frac{k(k+1)}{2}+k+1$

$k+k+1=\frac{k(k+1)}{2}+\frac{k+1}{1}\cdot 2$

$k+k+1=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$

$k+k+1=\frac{k^2+k}{2}+\frac{2k+2}{2}$

$k+k+1=\frac{k^2+3k+2}{2}$

$k+k+1=\frac{(k+1)(k+2)}{2}$

Lesson 5: The Pell Sequence

Exercises:

1) The Pell Sequence is defined by $p_0 = 1$, $p_1 = 2$ and $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$.

a) Find the first 6 terms of the sequence

1, 2, 5, 12, 29, 70

$2(2)+1=5$
$2(5)+2=12$
$2(12)+5=29$
etc.

b) Use the characteristic polynomial technique to solve this recurrence relation.

$P_n - 2P_{n-1} - P_{n-2} = 0$

$x^2 - 2x - 1 = 0$

$x_1 = 1+\sqrt{2}$

$x_2 = 1-\sqrt{2}$

$x = \frac{2 \pm \sqrt{4-4(-1)}}{2}$

$x = \frac{2 \pm \sqrt{8}}{2}$

$x = \frac{2 \pm 2\sqrt{2}}{2}$

$x = 1 \pm \sqrt{2}$

$1+\sqrt{2} = (1+\sqrt{2})C_1 + (1+\sqrt{2})C_2$

$(-)\quad 2 = (1+\sqrt{2})C_1 + (1-\sqrt{2})C_2$

$\sqrt{2} - 1 = (2\sqrt{2})C_2$

$\frac{2-\sqrt{2}}{4} = C_2$

$\frac{\sqrt{2}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$\frac{2-\sqrt{2}}{4}$

$P_n = C_1(1+\sqrt{2})^n + C_2(1-\sqrt{2})^n$

$P_0 = C_1(1+\sqrt{2})^0 + C_2(1-\sqrt{2})^0$

$1 = C_1 + C_2 \xrightarrow{\times 1\sqrt{2}}$

$1 = C_1 + \frac{2-\sqrt{2}}{4}$

$-\left(\frac{2-\sqrt{2}}{4}\right) \quad -\left(\frac{2-\sqrt{2}}{4}\right)$

$\frac{2+\sqrt{2}}{4} = C_1$

$P_1 = C_1(1+\sqrt{2})^1 + C_2(1-\sqrt{2})^1$

$2 = (1+\sqrt{2})C_1 + C_2(1-\sqrt{2})$

$P_n = \frac{2+\sqrt{2}}{4}(1+\sqrt{2})^n + \frac{2-\sqrt{2}}{4}(1-\sqrt{2})^n$

2) There exist closed form solutions for p_n (Pell Sequence). Below is one example. Calculate the first 6 terms of the sequence by using the formula below.

$$p_n = \sum_{\substack{i,j,k \ge 0 \\ i+j+2k=n}} \frac{(i+j+k)!}{i!\,j!\,k!}$$

$P_n \rightarrow i + j + 2K = n$

$P_0 \rightarrow i + j + 2K = 0$

$\underset{i}{(0}, \underset{j}{0}, \underset{K}{0})$

$$\frac{(i+j+K)!}{i!\,j!\,K!} = \frac{(0+0+0)!}{0!\,0!\,0!} = \frac{0!}{1 \cdot 1 \cdot 1} = \frac{1}{1} = \overset{P_0}{\boxed{1 \checkmark}}$$

$P_1 \rightarrow i + j + 2K = 1$

$(0, 1, 0)$

$+ \ (1, 0, 0)$

$$\frac{(0+1+0)!}{0!\,1!\,0!} + \frac{(1+0+1)!}{1!\,0!\,0!}$$

$$1 + 1 = \overset{P_1}{\boxed{2 \checkmark}}$$

$P_2 = 5$ true?

$P_2 \rightarrow i + j + 2K = 2$

$(1, 1, 0)$

$(0, 0, 1)$

$+$

$(2, 0\ 0)$

$(0, 2, 0)$

$$\frac{(1+1+0)!}{1!\,1!\,0!} + \frac{(0+0+1)!}{0!\,0!\,1!} + \frac{(2+0+0)!}{2!\,0!\,0!} + \frac{(0+2+0)!}{0!\,2!\,0!}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{2}{1} \qquad \frac{1}{1} \qquad \frac{2}{2} \qquad \frac{2}{2}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$2 + 1 + 1 + 1$$

$P_2 \boxed{= 5\ \checkmark}$

* = 6

lower the #
smaller the dish

move

	A	B	C
start	1,2	*	*
1	2	1	
2	*	1	2
3	*	*	1,2

n=2, moves=3

	A	B	C
START	1,2,3	*	*
1	2,3	*	1
2	3	2	1
3	3	1,2	*
4	*	1,2	3
5	1	2	3
6	1	*	3,2
7	*	*	1,2,3

$a_0 \; a_1 \; a_2 \; a_3 \; a_4 \; a_5$

$0, 1, 3, 7, 15, 31$

$+1 \; +2 \; +4 \; +8 \; +16$

B. $a_n = 2a_{n-1} + 1$
recurrence relation

C. $a_n = 2^n - 1$
explicit formula

	A	B	C
START	1,2,3,4	*	*
1	2,3,4	1	*
2	3,4	1	2
3	3,4	*	1,2
4	4	3	1,2
5	4	3,2	1
6	4	3,2,1	*
7	*	1,2,3	4
8	*	2,3	1,4
9	2	3	1,4
10	1,2	3	4
11	1,2	*	3,4
12	2	1	3,4
13	*	1	2,3,4
14	*	*	1,2,3,4
15			

really 15
but I skipped
a step here